Fuzzy Topology On Fuzzy Sets: Regularity and Separation Axioms

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Abstract:
In this paper, separation and regularity axioms in fuzzy topology on fuzzy set are defined and studied. We investigate some of its characterizations and discuss certain relationship among them with some necessary counterexamples. Moreover some of their basic properties are examined. In addition, goodness and hereditary properties are discussed.

1. Introduction:

The notion of fuzzy topology on fuzzy sets was introduced by Chakraborty and Ahsanullah [1] as one of treatments of the problem which may be called the subspace problem in fuzzy topological spaces. One of the advantages of defining topology on a fuzzy set lies in the fact that subspace topologies can now be developed on fuzzy subsets of a fuzzy set. Later Chaudhury and Das [2] studied several fundamental properties of such fuzzy topologies. The concept of separation axioms is one of most important concepts in topology. In fuzzy setting, it had been studied by many authors such as [3,5,6,7,10].However, the separation and regularity axioms has not yet been studied in the new setting, only in [2] they introduced the concept of Hausdorff, regular and normal spaces. The object of the present paper is to introduce a set of new regularity and separation axioms which are called (FR_i ,i = 0,1,2,3) and (FT_i ,i= 0,1,2,3,4) by using quasi-coincident and neighborhood system. Our work organized as follows, In section 1. We give some preliminary concepts, investigating some of new results in the new setting. In section 2. We give the definition of regularity axioms (FR_i ; i=0,1,2,3) and some characteristics theorems are proved. Next the separation axioms (FT_i; i = 0,1,2,3,4) are introduced, investigating many of its properties in section 3. Finally, in section 4. We examine the hereditary and good extension property in the sense of Lowen [9].

2. Definitions and Notations

Throughout this paper, X denotes a non-empty set, the symbol I will denote the closed unit interval and a fuzzy set A of X is a function with domain X and values in$I$. A fuzzy point x_α is a fuzzy set such $x_\alpha(y) = \alpha > 0$ if $x = y$ for all $y \in X$ and $x_\alpha(y) = 0$ if $x \neq y$. We write $x_\alpha \in A$ if $\alpha \leq A(x)$. The family of all fuzzy points of A will be denoted by $FP(A)$. If $A, B \in I^X$, $B(x) \leq A(x)$ $\forall\, x \in X$, then B is said to be a fuzzy subset of A and denoted by $B \subseteq A$. The family of all fuzzy subsets of A will denoted by $\mathcal{F}_A$ i.e $\mathcal{F}_A = \{B \in I^X : B \subseteq A\}$. The set $S(A) = \{x \in X : A(x) > 0\}$ is said to be the support of A. If $\alpha \in$ I, the fuzzy subset of X which assigns α $\forall$ x $\in$ X will be denoted by $\underline{\alpha}$. If $B \subset X$, then χ_B denotes the characteristic function of B on X.

1.1 Definition

If $B \subset S(A)$. Then$\chi_B^A = \chi_B \cap A$ denotes the characteristic function of B referred to A.
In general a fuzzy subset $E$of A is called a maximal if $E = \chi_{S(E)} \cap A$ i.e if $\forall x \in X, E(x) \neq 0$, then $E(x) = A(x)$. If $B \in \mathcal{F}_A$, then the complement of B referred to A, denoted by B'_A and defined by, $B'_A(x) = A(x) - B(x) \forall x \in X$. Let $U, V \in \mathcal{F}_A$. Then U, V are said to be quasi-coincident referred to A, denoted by $U q_A V$ iff there exists $x \in S(A)$ such that $U(x) + V(x) > A(x)$. If U is not quasi-coincident with Vreferred to A, then we denoted for this by $U \tilde{q}_A V$.

Now one can easily prove the following proposition as in [1].

1.2 Proposition

Let $U, V, G \in \mathcal{F}_A$ and x_α , $y_\beta \in FP(A)$. Then:
1) $U \tilde{q}_A V \Leftrightarrow U \subseteq V'_A$,
2) $U \tilde{q}_A V \Leftrightarrow V \tilde{q}_A U$,
3) $U \cap V = \underline{0} \Longrightarrow U \tilde{q}_A V$,
4) $U \tilde{q}_A U'_A$,
5) $U \tilde{q}_A V, G \subseteq V \subseteq A \Longrightarrow U \tilde{q}_A G$,

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6) $U \subseteq V \Leftrightarrow (x_\alpha q_A U \Longrightarrow x_\alpha q_A V), x_\alpha \in FP(A)$.
7) $x_\alpha q_A(\bigcup_{i\in J} U_i) \Leftrightarrow x_\alpha q_A U_i$ *, for some* $i \in J$,
8) $x_\alpha q_A(U \cap V) \Leftrightarrow (x_\alpha q_A U and x_\alpha q_A V)$,
9) $x \neq y \Longrightarrow x_\alpha \tilde{q}_A y_\beta \forall \alpha, \beta \in I$,
10) $x_\alpha \tilde{q}_A y_\beta \Leftrightarrow x \neq y or\ (x = y\ , \alpha + \beta \leq A(x))$.

1.3 Lemma
Let $U, V \in \mathcal{F}_A$ and $\{U_i: i \in J\} \subset \mathcal{F}_A$. Then:
i) $S(U \cap V) = S(U) \cap S(V)$,
ii) $S(\bigcup_{i\in J} U_i) = \bigcup_{i\in J} S(U_i)$.
Proof. Obvious.

Now we recall the basic definition of fuzzy topology on fuzzy set as in [1].

1.4 Definition
Let A be a fuzzy subset of X. A collection δ of fuzzy subsets of A i.e $\delta \subset \mathcal{F}_A$ satisfying the following conditions:
i) $\underline{0}$, $A \in \delta$,
ii) $U, V \in \delta \Longrightarrow U \cap V \in \delta$,
iii) $\{U_i: i \in J\} \subset \delta \Longrightarrow \bigcup_{i\in J} U_i \in \delta$,
is called a fuzzy topology on A. The pair (A, δ) is called a fuzzy topological space, members of δ will be called a fuzzy open sets and their complements referred to A are called a fuzzy closed sets of (A, δ). The family of all fuzzy closed sets in (A, δ) will be denoted by δ'_A.

Note: Unless otherwise mentioned by fuzzy topological spaces we shall mean it in the sense
of the above definition and (A, δ) will denote a fuzzy topological space.

1.5 Definition
A fuzzy topological space (A, δ) is called a fully stratified if each fuzzy subset in the form $\underline{\alpha} \cap A$ is in δ for all $\alpha \in I$.

1.6 Definition
Let (A, δ) be a fuzzy topological space, $x_\alpha \in FP(A)$. Then any fuzzy set $O_{x_\alpha} \in \delta$ contains x_α is called a neighborhood (nbd, for short) of x_α in (A, δ). The set of all neighborhoods of x_α will be denoted by, $N_A(x_\alpha)$. In general for any $B \in \mathcal{F}_A$, $O_B \in \delta$ denotes a fuzzy open subset of A contains B.

1.7 Definition
Let (A, δ) be a fuzzy topological space, $B \in \mathcal{F}_A$. Then the closure(interior) of B is defined by:
i) $\overline{B}_A = \cap \{U : U \in \delta'_A, B \subseteq U\}$,
ii) $B^\circ_A = \cup \{G : G \in \delta, G \subseteq B\}$, respectively.

1.8 Proposition
Let (A, δ) be a fuzzy topological space, $B \in \mathcal{F}_A$ and $x_\alpha \in FP(A)$. Then we have:
i) $(B^\circ_A)'_A = \left(\overline{B'_A}\right)_A$
ii) $x_\alpha \in B^\circ_A \Leftrightarrow$ there exists $O_{x_\alpha} \in N_A(x_\alpha)$ such that $O_{x_\alpha} \subseteq B$.
iii) $x_\alpha q_A \overline{B} \Leftrightarrow O_{x_\alpha} q_A B$, for all $O_{x_\alpha} \in N_A(x_\alpha)$.
iv) $V q_A B \Leftrightarrow V q_A \overline{B}$, for all $V \in \delta$.
Proof. Stratiforward.

In the following we recall the concept of the strong α-cut of any fuzzy subset of A as in [10].
1.9 Definition: For any $B \in \mathcal{F}_A$. We define, $B_\alpha = \{x \in X: B(x) > \alpha\}$, $\alpha \in I \backslash \{1\}$.

1.10 Proposition
Let $\{B_i: i \in J\} \subset \mathcal{F}_A$, $\alpha \in I \backslash \{1\}$ and S is a finite index set. Then we have:
i) $(\bigcup_{i\in J} B_i)_\alpha = \bigcup_{i\in J} (B_i)_\alpha$,
ii) $(\bigcap_{i\in S} B_i)_\alpha \subseteq \bigcap_{i\in S} (B_i)_\alpha$.

By using the Lemma (1.3), it is easy to prove the following theorem.

1.11 Theorem

a) Let $\underline{0} \neq A \in I^X$ and $(S(A), \tau)$ be a topological space on $S(A)$. Then the following structures:

$i)$ $\delta_\tau = \{B \in \mathcal{F}_A : S(B) \in \tau\}$ and,

$ii)$ $\Delta_\tau = \{\chi_B^A : B \in \tau\}$,

are fuzzy topologies on A generated by τ.

b) Let (A, δ) be a fuzzy topological space on A. Then the following structures:

i) $\tau_\delta = \{S(B) : B \in \delta\}$ and,

ii) $[\delta] = \{B \subseteq S(A) : \chi_B^A \in \delta\}$,

are ordinary topologies on $S(A)$ generated by δ.

1.12 Proposition

Let $\underline{0} \neq A \in I^X$, $(S(A), \tau)$ be a topological space on $S(A)$ and (A, δ) a fuzzy topological space on A. Then:

i) $\underline{\alpha} \cap A \in \delta_\tau \ \forall \alpha \in I$,

ii) $\forall B \in \tau$, $\chi_B^A \in \delta_\tau$ and then $\Delta_\tau \leq \delta_\tau$,

iii) $\tau = \tau_{\delta_\tau}$ and $\delta \leq \delta_{\tau_\delta}$.

Proof. Straightforward.

1.13 Definition

Let $A \in I^X$. A topological space $(S(A), \tau)$ is said to be an s-topological space on $S(A)$ iff, τ contains A_α for all $\alpha \in I$.

The following example shows the existence of s-topological space and shows that a topological space $(S(A),\tau)$ need not be s-topological space and shows that the family, $\tau \cup \{A_\alpha : \alpha \in I\}$ need not be a topological space on S(A).

1.14 Example

Let $X = \{x, y, z\}$ and $A = (x_{0.5}, y_1, z_{0.5}) \in I^X$. Then we have:

i) $S(A) = X$ and $\{A_\alpha : \alpha \in I\} = \{X, \{y\}\}$.

ii) $\tau_1 = \{\phi, X, \{x\}, \{x, z\}\}$ is a topology on $S(A)$ but not s-topology.

iii) $\tau_2 = \{\phi, X, \{x\}, \{y\}, \{x, y\}\}$ is s-topology on $S(A)$.

iv) $\tau_1 \cup \{A_\alpha : \alpha \in I\} = \{\phi, X, \{x\}, \{y\}, \{x, z\}\}$ is not a topology on $S(A)$.

1.15 Definition

Let $(S(A), \tau)$ be an s-topological space on $S(A)$. Then a fuzzy subset B of A is said to be a fuzzy lower semi-continuous function on $S(A)$ if $B_\alpha \in \tau$ for all $\alpha \in I$. The set of all fuzzy subsets of A which is lower semi-continuous functions on $S(A)$ will be denoted by, $\omega_A(\tau)$ *i.e.* $\omega_A(\tau) = \{B \in \mathcal{F}_A : B_\alpha \in \tau, \alpha \in I\}$.

1.16 Proposition

Let $(S(A), \tau)$ be an s-topological space. Then the family $\omega_A(\tau)$ is a fuzzy topology on A and $(A, \omega_A(\tau))$ is called an induced fuzzy topological space by τ.

Now one can easily prove the following lemma.

1.17 Lemma

Let $(S(A), \tau)$ be an s-topological space on $S(A)$, $B \leq A$ and $G \subset S(A)$. Then we have:

i) $B \in \omega_A(\tau)$ *iff* $(B_\alpha \in \tau \forall \alpha \in I \backslash \{1\})$,

ii) $G \in \tau$ *iff* $\chi_G^A \in \omega_A(\tau)$,

iii) $\underline{\alpha} \cap A \in \omega_A(\tau)$ *for all* $\alpha \in I$,

iv) $\overline{(\chi_G^A)}_A = \chi_{\bar{G}}^A$,

v) $B = \bigcup_{\alpha \in I} \alpha \chi_{B_\alpha}^A$.

1.18 Definition

A fuzzy topological space (A, δ) is said to be a weakly induced iff, for every $B \in \delta$, $B_\alpha \in [\delta]$ for all $\alpha \in I$ i.e. iff every element in δ is a fuzzy lower semi-continuous function from $(S(A), [\delta])$ to I.

Note: One of the advantages of defining topology on a fuzzy set lies in the fact that subspace topologies can now be developed on fuzzy subsets of a fuzzy set as follows:

1.19 Definition[1,2]

Let (A, δ) be a fuzzy topological space and $Y \subseteq A$. Then the family $\delta_Y = \{Y \cap V : V \in \delta\}$ is a fuzzy topology on Y and (Y, δ_Y) is called a fuzzy subspace of (A, δ).

Note: If Y is a maximal subset of A, then (Y, δ_Y) is called a maximal subspace of (A, δ) .

1.20 Proposition[13,15]
Let (Y, δ_Y) be the maximal subspace of a FTS (A, δ). Then:
i) $E \subset Y$is closed in (Y, δ_Y) if and only if $E = Y \cap B$, where $B \in \delta'_A$,
ii) For every $E \subseteq Y$, we have $\overline{E}_Y = Y \cap \overline{E}_A$, where $\overline{E}_Y, \overline{E}_A$ are closures of E in (Y, δ_Y) and (A, δ), respectively.

2. Fuzzy regularity axioms

2.1 Definition
A fuzzy topological space (A, δ) is said to be:
i) FR_0-space iff $\forall x_\alpha , y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A \overline{y}_\beta$ implies $\overline{x}_\alpha \tilde{q}_A y_\beta$,
ii) FR_1-space iff $\forall x_\alpha , y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A \overline{y}_\beta$ implies there exist $O_{x_\alpha} \in N_A(x_\alpha)$ and $O_{y_\beta} \in N_A(y_\beta)$ such that $O_{x_\alpha} \tilde{q}_A O_{y_\beta}$.
iii) FR_2-space iff ($\forall x_\alpha \in FP(A)$ and $\forall B \in \delta'_A$) with $x_\alpha \tilde{q}_A B$ implies there exist $O_{x_\alpha} \in N_A(x_\alpha)$ and $O_B \in N_A(B)$ such that $O_{x_\alpha} \tilde{q}_A O_B$.
iv) FR_3-space iff $\forall U, V \in \delta'_A$with $U \tilde{q}_A V$ implies there exist$O_U \in N_A(U)$ and $O_V \in N_A(V)$ such that $O_U \tilde{q}_A O_V$.

Note: FR_2 (resp. FR_3) spaces are those which are called fuzzy regular (resp. fuzzy normal) spaces and was introduced in [2] as an extension of its original concept in [8].

In the following we introduce some properties of FR_0 space.
2.2 Theorem
Let (A, δ) be a fuzzy topological spaces, $x_\alpha \in FP(A)$ and $F \in \delta'_A$. Then the following statements are equivalent:
1) (A, δ) is a FR_0-space,
2) $\overline{x}_\alpha \subseteq O_{x_\alpha}, \forall O_{x_\alpha} \in N_A(x_\alpha)$.
3) $\overline{x}_\alpha \subseteq \cap \{O_{x_\alpha} : O_{x_\alpha} \in N_A(x_\alpha)\}$.
4) $x_\alpha \tilde{q}_A F$ implies there exists $O_F \in N(F)$ such that $x_\alpha \tilde{q}_A O_F$.
5) $x_\alpha \tilde{q}_A F$ implies $\overline{x}_\alpha \tilde{q}_A F$.
6) $x_\alpha \tilde{q}_A y_\beta$ implies $\overline{x}_\alpha \tilde{q}_A \overline{y}_\beta$.
Proof.
1) $\Longrightarrow$ 2) Let $y_\beta q_A \overline{x}_\alpha \overset{(1)}{\Rightarrow} x_\alpha q_A \overline{y}_\beta$. By (ii) of Proposition (1.8) we have $y_\beta q_A O_{x_\alpha} \forall O_{x_\alpha} \Rightarrow \overline{x}_\alpha \subseteq O_{x_\alpha} \forall O_{x_\alpha} \in N_A(x_\alpha)$ (by (6) of Proposition (1.2)).
2) $\Longrightarrow$3) is obvious.
3) $\Longrightarrow$ 4) Let $x_\alpha \tilde{q}_A F \Longrightarrow x_\alpha \in F'_A \overset{(2)}{\Rightarrow} \overline{x}_\alpha \subseteq F'_A \Longrightarrow F \subseteq \overline{x}'_\alpha = O_F$, $so x_\alpha \tilde{q}_A \overline{x}'_\alpha = O_F$.
4) $\Longrightarrow$ 5) Let $x_\alpha \tilde{q}_A F \overset{(4)}{\Rightarrow}$ there exists O_F such that$x_\alpha \tilde{q}_A O_F \Longrightarrow x_\alpha \in O'_F \Longrightarrow \overline{x}_\alpha \subseteq O'_F \Longrightarrow$ $\overline{x}_\alpha \tilde{q}_A O_F \Longrightarrow \overline{x}_\alpha \tilde{q}_A F$.
5) $\Longrightarrow$ 6) and 6) $\Longrightarrow$ 1) are obvious.

2.3 Theorem
The following implications hold:

$$FR_3 \wedge FR_0 \Longrightarrow FR_2 \Longrightarrow FR_1 \Longrightarrow FR_0.$$

Proof
i) Let (A, δ) be a FR_3, FR_0 and let $x_\alpha \tilde{q}_A G$, $G \in \delta'$. Then from (5) of the above theorem we have $\bar{x}_\alpha \tilde{q}_A G$. Since (A, δ) is FR_3, then there exist $O_{\bar{x}_\alpha}$, O_G such that $O_{\bar{x}_\alpha} \tilde{q}_A O_G$. Take , $O_{x_\alpha} = O_{\bar{x}_\alpha}$, then $O_{x_\alpha} \tilde{q}_A O_G$ and hence (A, δ) is FR_2-space.
ii) Let (A, δ) be a FR_2 space and let $x_\alpha \tilde{q}_A \bar{y}_\beta$. Then there exist O_{x_α} and $O_{\bar{y}_\beta} \in \delta$ such that $O_{x_\alpha} \tilde{q}_A O_{\bar{y}_\beta}$. Take $O_{y_\beta} = O_{\bar{y}_\beta} \Rightarrow O_{x_\alpha} \tilde{q}_A O_{y_\beta}$. Hence (L^X, δ) is a FR_1 space.
iii) Let (A, δ) be a FR_1space, $y_\beta \tilde{q}_A O_{x_\alpha} \Rightarrow y_\beta \subseteq O'_{x_\alpha} \Rightarrow \bar{y}_\beta \subseteq O'_{x_\alpha} \Rightarrow O_{x_\alpha} \tilde{q}_A \bar{y}_\beta \Rightarrow x_\alpha \tilde{q}_A \bar{y}_\beta$ and so there exist $O^*_{x_\alpha}$, $O_{\bar{y}_\beta} \in \delta$ such that $O^*_{x_\alpha} \tilde{q}_A O_{y_\beta} \Rightarrow y_\beta \tilde{q}_A \bar{x}_\alpha$ (by (iii) of Proposition (1.8)). So by (6) of Proposition (1.2)) we get $\bar{x}_\alpha \subseteq O_{x_\alpha}$ $\forall O_{x_\alpha}$.Hence (A, δ) is FR_0.
2.4 Corollary

Let(A,δ)be a fuzzy topological space. Then(A,δ) is a FR$_1$ if and only if $\forall\ x_\alpha, y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A \overline{y}_\beta$ implies there exist $O_{\overline{x}_\alpha},\ O_{\overline{y}_\beta} \in \delta$ such that $O_{\overline{x}_\alpha} \tilde{q}_A O_{\overline{y}_\beta}$.

Proof. Follows from the above implication and from (2) of Theorem (2.2).

2.5 Lemma

Let $(S(A),\tau)$be a topological space, $x_\alpha \in A$. Then we have:

i) $x_\alpha \in \delta_\tau'$ for all $0 \neq \alpha < A(x)$,

ii) $\overline{\chi^A_{\{x\}}}^{\delta_\tau} = \chi^A_{\overline{\{x\}}^\tau}\ \ forall\ x \in S(A)$.

Proof. Obvious.

2.6 Theorem

Let $(S(A),\tau)$ be a topological space on $S(A)$. Then (A,δ_τ) is FR$_0$ if and only if $(S(A),\tau)$ is a R$_0$-space.

Proof. Let (A,δ_τ) be a FR$_0$-space $x \in \overline{y}$. Then $x_{A(x)} q_A \overline{y}_{A(y)}$(by the above lemma) implies $x_{A(x)} q_A O_{y_{A(y)}} (since (A,\delta_\tau)\ is\ FR_0) \Longrightarrow y_{A(y)} q_A \overline{x}_{A(x)}$ (by (iii) of Proposition (1.8)) $\Longrightarrow y \in \overline{x}$ (by the above lemma). Hence$(S(A),\tau)$ is a R_0 -space.

Conversely, let $(S(A),\tau)$ be a R$_0$ -space, $x_\alpha \in A$. Since $x_\alpha \in \delta_\tau'\ \ \forall \alpha < A(x)$. Then $\overline{x}_\alpha = x_\alpha \subseteq O_{x_\alpha} \forall O_{x_\alpha}$, when $\alpha = A(x)$ then clearly $\overline{x}_{A(x)} = O_{x_{A(x)}}$ (since $(S(A),\tau)$ is R_0 *i.e.* $\overline{x} \subseteq O_x\ \forall\ O_x \in \tau$). Hence (A,δ_τ) is a FR$_0$-space.

2.7 Theorem

Let $(S(A),\tau)$ be a topological space. If $(S(A),\tau)$is R_1 -space, then (A,δ_τ) is FR$_1$-space.

Proof. Let $(S(A),\tau)$ be a R$_1$-space, $x_\alpha, y_\beta \in FP(A)$ *with* $x_\alpha \tilde{q}_A \overline{y_\beta}$. Then either $x \neq y$ or ($x = y$, $\alpha + \beta \leq A(x)$).

(a) If $x \neq y$, then either $\overline{x} \neq \overline{y}$ or $\overline{x} = \overline{y}$.

i) If $\overline{x} \neq \overline{y}$, then there exist $O_x, O_y \in \tau$ such that $O_x \cap O_y = \phi$. Now we take $O_{x_\alpha} = \chi^A_{O_x} \in \delta_\tau$ and $O_{y_\beta} = \chi^A_{O_y} \in \delta_\tau$, then $\chi_{O_x} \tilde{q}_A O_{y_\beta}$. Hence (A,δ_τ) is FR$_1$ space.

ii) If $x \neq y$, $\overline{x} = \overline{y}$, this case is excluded (since (X,τ) is R_1).

(b) If $x = y$, $\alpha + \beta \leq A(x)$, then we take $O_{x_\alpha} = \underline{\alpha} \cap A$, $O_{y_\beta} = \underline{\beta} \cap A \in \delta_\tau$ to be the required neighborhoods. Hence (A,δ_τ) is a FR$_1$-space.

2.8 Theorem

Let (A,δ) be a fuzzy topological space. Then $(A,\delta) is\ FR_2$ if and only if for all $x_\alpha \in FP(A)$ and for all $O_{x_\alpha} \in N(x_\alpha)$ there exists $O^*_{x_\alpha}$ such that $\overline{O^*_{x_\alpha}} \subseteq O_{x_\alpha}$.

Proof. Let (A,δ) be a FR$_2$, $x_\alpha \in FP(A)$ and $O_{x_\alpha} \in N_A(x_\alpha)$. Then $x_\alpha \tilde{q}_A O'_{x_\alpha} \Longrightarrow$ there exist $O^*_{x_\alpha} \in N(x_\alpha)$, $G \in N(O'_{x_\alpha})$ such that $O^*_{x_\alpha} \tilde{q}_A G$ implies that $O^*_{x_\alpha} \subseteq G'_A \in \delta'_A \Longrightarrow \overline{O^*_{x_\alpha}} \subseteq G'_A \subseteq O_{x_\alpha}$.

Conversely, let $x_\alpha \in FP(A)$, $G \in \delta'_A$ be such that $x_\alpha \tilde{q}_A\ G$. Then $x_\alpha \subseteq G'_A$ i.e. $G'_A \in N(x_\alpha)$, so there exists $O^*_{x_\alpha}$ such that $\overline{O^*_{x_\alpha}} \subseteq O_{x_\alpha} = G'_A$ (by hypothesis)$\Rightarrow G \subseteq (\overline{O^*_{x_\alpha}})' \subseteq = O_G$ and $O_G \tilde{q}_A O^*_{x_\alpha}$. Hence (A,δ) is a FR$_2$-space.

2.9 Theorem

Let (A,δ) be a fuzzy topological space . Then (A,δ) is FR_3(normal) if and only if $\forall\ F \in \delta'$, $\forall\ O_F$ there exists O^*_F such that $\overline{O^*_F} \subseteq O_F$.

Proof. The proof is analogous to the above proof.

3. Fuzzy separation axioms

3.1 Definition

A fuzzy topological space (A,δ) is said to be:

i) FT$_0$-space iff $\forall x_\alpha, y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A y_\beta$ implies there exists $O_{x_\alpha} \in N_A(x_\alpha)$ such that $O_{x_\alpha} \tilde{q}_A y_\beta$ or there exists $O_{y_\beta} \in N_A(y_\beta)$ such that $O_{y_\beta} \tilde{q}_A x_\alpha$.

ii) FT$_1$-space iff $\forall x_\alpha, y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A y_\beta$ implies there exist $O_{x_\alpha} \in N_A(x_\alpha)$ such that $O_{x_\alpha} \tilde{q}_A y_\beta$ and there exists $O_{y_\beta} \in N_A(y_\beta)$ such that $O_{y_\beta} \tilde{q}_A x_\alpha$.

iii) FT$_2$-space iff $\forall x_\alpha, y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A y_\beta$ implies thereexist $O_{x_\alpha} \in N_A(x_\alpha)$ and $O_{y_\beta} \in N_A(y_\beta)$ such that $O_{x_\alpha} \tilde{q}_A O_{y_\beta}$.

iv) FT$_3$-space iff it is FR$_2$ and FT$_1$-space.

v) FT$_4$-space iff it is FR$_3$ and FT$_1$-space.

3.2 Theorem

Let (A,δ) be a fuzzy topological space. Then (A,δ) *is* FT_0 if and only if ($x_\alpha \tilde{q}_A y_\beta$ implies $x_\alpha \tilde{q}_A \overline{y}_\beta$ or $\overline{x}_\alpha \tilde{q}_A y_\beta$).

Proof. Let (A,δ) *be a* FT_0 and $x_\alpha, y_\beta \in FP(A)$ with $x_\alpha \tilde{q}_A y_\beta$. Then there exists O_{x_α} such that $O_{x_\alpha} \tilde{q}_A y_\beta \Rightarrow x_\alpha \tilde{q}_A \bar{y}_\beta$ or there exists O_{y_β} such that $x_\alpha \tilde{q}_A O_{y_\beta} \Rightarrow \bar{x}_\alpha \tilde{q}_A y_\beta$ (by (iii) of Proposition (1.8)).Conversely, let $x_\alpha \tilde{q}_A \bar{y}_\beta$ or $\bar{x}_\alpha \tilde{q}_A y_\beta \Rightarrow x_\alpha \le \bar{y}'_\beta = y'^{\circ}_\beta\ \tilde{q}_A y_\beta$ or $y_\beta \le \bar{x}'_\alpha = x'^{\circ}_\alpha \tilde{q}_A x_\alpha$.Hence (A,δ) is a FT$_0$ space.

3.3 Theorem

Let $(S(A),\tau)$be a topological space. Then (A,δ_τ) is FT_0 if and only if $(S(A),\tau)$ is T_0-space.

Proof

Let (A,δ_τ) be a FT$_0$-space and $x \ne y$. Then $x_\alpha \tilde{q}_A y_\beta$, in particular $x_{A(x)} \tilde{q}_A y_{A(y)}$ implies there exists $O_{x_{A(x)}} \in \delta_\tau$ such that $y_{A(y)} \tilde{q}_A O_{x_{A(x)}}$ or there exists $O_{y_{A(y)}} \in \delta_\tau$ such that $x_{A(x)} \tilde{q}_A O_{y_{A(y)}}$.

Take $O_x = S(O_{x_{A(x)}}) \in \tau$, then $y \notin S(O_{x_{A(x)}})$ or take $O_y = S(O_{y_{A(y)}}) \in \tau$, then $x \notin S\left(O_{y_{A(y)}}\right)$. Hence $(S(A),\tau)$ is a T_0-space.

Conversely, let $(S(A),\tau)$ be a T_0 and $\forall x_\alpha, y_\beta \in FP(A)$ *with* $x_\alpha \tilde{q}_A y_\beta$. Then either $x \ne y$ or ($x = y$, $\alpha+\beta \le A(x)$). If $x \ne y$, then there exists $O_x \in \tau$ such that $y \notin O_x$ or there exists $O_y \in \tau$ such that $x \notin O_y$. Now take $O_{x_\alpha} = \chi^A_{O_x} \in \delta_\tau$, then $y_\beta \tilde{q}_A \chi^A_{O_x}$ or take $O_{y_\beta} = \chi^A_{O_y} \in \delta_\tau$, then $x_\alpha \tilde{q}_A \chi^A_{O_y}$. Hence (A,δ_τ) is a FT$_0$-space.

If $x = y$, $\alpha+\beta \le A(x)$, then we take $O_{x_\alpha} = \underline{\alpha} \cap A \in \delta_\tau$ and $O_{y_\beta} = \underline{\beta} \cap A \in \delta_\tau$ (by(i) of Proposition (1.12)) to be the required neighborhoods. Hence (A,δ_τ) is a FT_0-space.

3.4 Theorem

Let (A,δ) be a fuzzy topological spaces. If (A,δ) *is* FT_0 , then $(S(A),\tau_\delta)$ is a T_0-space.

Proof. The proof is analogous to that of necessity of the above theorem.

3.5 Theorem

Let (A,δ) be a fully stratified fuzzy topological space. Then (A,δ) *is* FT_0 if and only if $(S(A),\tau_\delta)$ is a T_0-space.

Proof. Necessity, follows from the above theorem.

Conversely, let $(S(A),\tau_\delta)$ be a T_0-space and $x_\alpha \tilde{q}_A y_\beta$ $\forall x_\alpha, y_\beta \in FP(A)$. Then either $x \ne y$ or ($x = y$, $\alpha+\beta \le A(x)$). If $x \ne y$, then there exists $O_x = S(\chi^A_{O_x}) \in \tau_\delta$ such that $y \notin O_x$ or there exists $O_y = S\left(\chi^A_{O_y}\right) \in \tau_\delta$ such that $x \notin O_y$. Now take $O_{x_\alpha} = \chi^A_{O_x} \in \delta$, then $y_\beta \tilde{q}_A \chi^A_{O_x}$ or take $O_{y_\beta} = \chi^A_{O_y} \in \delta$, then $x_\alpha \tilde{q}_A \chi^A_{O_y}$. Hence (A,δ_τ) is a FT$_0$-space.

If ($x = y$, $\alpha+\beta \le A(x)$), then Take $O_{x_\alpha} = \underline{\alpha} \cap A \in \delta$ or $O_{y_\beta} = \underline{\beta} \cap A \in \delta$ (since (A,δ) is fully stratified) to be the required neighborhoods. Hence (A,δ) is a FT_0-space.

3.6 Theorem

Let (A,δ) be a fully stratified fuzzy topological space. If $(S(A),[\delta])$ is a T_0-space, then (A,δ) *is a* FT_0 -space.

Proof. Let $(S(A),[\delta])$be a T_0 and $x_\alpha \tilde{q}_A y_\beta$, $\forall x_\alpha, y_\beta \in FP(A)$. Then either $x \ne y$ or ($x = y$, $\alpha+\beta \le A(x)$). If $x \ne y$, then there exists $O_x \in [\delta]$ such that $y \notin O_x$ or there exists $O_y \in [\delta]$ such that $x \notin O_y$. Now take $O_{x_\alpha} = \chi^A_{O_x} \in \delta$, then $y_\beta \tilde{q}_A \chi^A_{O_x}$ or take $O_{y_\beta} = \chi^A_{O_y} \in \delta$, then $x_\alpha \tilde{q}_A \chi^A_{O_y}$. Hence (A,δ) *is* FT_0 .

If $x = y$, $\alpha+\beta \le A(x)$, then take $O_{x_\alpha} = \underline{\alpha} \cap A \in \delta$ or $O_{y_\beta} = \underline{\beta} \cap A \in \delta$ (since (A,δ) is fully stratified) to be the required neighborhoods. Hence (A,δ) is a FT_0-space.

3.7 Theorem

Let (A,δ) be a fully stratified and weakly induced fuzzy topological space. Then $(S(A),[\delta])$ is T_0-space if and only if (A,δ) *is* FT_0 space.

Proof. Necessity, follows from the above theorem.

Conversely, let (A,δ) *be* FT_0 and $x = y$, then $x_{A(x)} \tilde{q}_A y_{A(y)}$. Since (A,δ) *is* FT_0 , then there exists $O_{x_{A(x)}} \in \delta$ such that $y_{A(y)} \tilde{q}_A O_{x_{A(x)}}$or there exists $O_{y_{A(y)}} \in \delta$ such that $x_{A(x)} \tilde{q}_A O_{y_{A(y)}}$. Now take $O_x = (O_{x_{A(x)}})_\alpha \in [\delta]$ or take $O_y = (O_{y_{A(y)}})_\alpha \in [\delta]$ (since (A,δ) is weakly induced), then it is easy to see that $y \notin O_x$ or $x \notin O_y$. Hence $(S(A),[\delta])$ is T_0-space.

In the following theorems we study some properties of FT_1 spaces.

3.8 Theorem

Let (A,δ) be a fuzzy topological space. Then the following statements are equivalent:

i) (A,δ) is a FT$_1$ space,

ii) $\forall\, x_\alpha, y_\beta \in FP(A)$ with $x_\alpha\, \tilde{q}_A\, y_\beta$ implies $x_\alpha\, \tilde{q}_A\, \bar{y}_\beta$ and $y_\beta\, \tilde{q}_A\, \bar{x}_\alpha$,
iii) $\bar{x}_\alpha = x_\alpha$, $\forall\, x_\alpha \in FP(A)$.
Proof. i) $\Leftrightarrow$ ii) is clearly from (iii) of Proposition (1.8).
i) $\Rightarrow$iii) Let $x_\alpha\, \tilde{q}_A\, y_\beta \Rightarrow$ there exists O_{y_β} such that $x_\alpha\, \tilde{q}_A\, O_{y_\beta}$ this implies $O_{y_\beta} \subseteq (x_\alpha)'_A$, thus $(x_\alpha)'_A$ is open i.e. x_α is closed $\Rightarrow \bar{x}_\alpha = x_\alpha$. And this is true for every $x_\alpha \in FP(A)$.
iii)$\Rightarrow$ i) $\bar{x}_\alpha = x_\alpha\ \forall\, x_\alpha \in FP(A)$ and $x_\alpha\, \tilde{q}_A\, y_\beta$. Then $x_\alpha, y_\beta \in \delta'_A$. Since $y_\beta\, \tilde{q}_A\, y'_\beta = O_{x_\alpha}$ and $x_\alpha\, \tilde{q}\, x'_\alpha = O_{y_\beta}$. Hence (A,δ) is a FT_1-space.

3.9 Theorem

Let (A,δ) be a fuzzy topological space. If (A,δ) *is* FT_1, then $(S(A),\tau_\delta)$ is a T_1 space.
Proof. Let (A,δ) be a FT_1 and $x \in S(A)$. Then $\bar{x}_{A(x)} = x_{A(x)}$ and so $x'_{A(x)} \in \delta$ this implies that $S(\,x'_{A(x)}) = S(A)\setminus\{x\} \in \tau_\delta$ so $\{x\}$ is closed for all $x \in S(A)$. Hence $(S(A),\tau_\delta)$ is a T_1 space.

3.10 Theorem

Let (A,δ) be a fully stratified fuzzy topological space. Then (A,δ) *is* FT_1 if and only if $(S(A),\tau_\delta)$ is T_1-space.
Proof. Necessity, follows from the above theorem. Conversely, the proof is similar to that of Theorem (3.5).

3.11 Theorem

Let $(S(A),\tau)$ be a topological space. Then $(S(A),\tau)$ is T_1 if and only if (A,δ_τ) is FT_1.
Proof. Necessity, let $(S(A),\tau)$ be a T_1-space and $x_\alpha \tilde{q}\, y_\beta\ \forall\, x_\alpha, y_\beta \in FP(A)$. Then either $x \neq y$ or ($x = y$, $\alpha+\beta \leq A(x)$). Now if $x \neq y$, then there exists $O_x \in \tau$ such that $y \notin O_x$ and there exists $O_y \in \tau$ such that $x \notin O_y$. Take $O_{x_\alpha} = \chi^A_{O_x} \in \delta_\tau$ and $O_{y_\beta} = \chi^A_{O_y} \in \delta_\tau$, then $y_\beta \tilde{q}_A \chi^A_{O_x}$ and $x_\alpha\, \tilde{q}_A \chi^A_{O_y}$. Hence (A,δ_τ) is FT_1 space.
If ($x = y$, $\alpha+\beta \leq A(x)$). Take $O_{x_\alpha} = \underline{\alpha} \cap A \in \delta_\tau$ and $O_{y_\beta} = \underline{\beta} \cap A \in \delta_\tau$ to be the required neighborhoods. Hence (A,δ_τ) is a FT_1- space.
Conversely, let (A,δ_τ) be a FT_1-space Then $(S(A),\tau)$ is a T_1–space (by Theorem (3.10)). But $\tau = \tau_{\delta_\tau}$ (by (iii) of Proposition (1.12)). Hence (X,τ) is a T_1-space.

3.12 Theorem

Let (A,δ) be a fully stratified fuzzy topological space. If $(S(A),[\delta])$ is T_1-space, then (A,δ) *is* FT_1 space.
Proof. The proof is analogous to that of Theorem (3.6).

3.13 Theorem

Let (A,δ) be a fully stratified and weakly induced fuzzy topological space. Then $(S(A),[\delta])$ is T_1-space if and only if (A,δ) *is* FT_1 space.
Proof. The proof is analogous to the proof of Theorem (3.7).

The following example shows that the converse of Theorem (3.4) and Theorem (3.9) may not be true in general.

3.14 Example

Let $X = \{x,y\}$, $A = (x_{0.5}, y_{0.5}) \in I^X$. Take $\delta = \{\underline{0}, A, (x_{0.5}, y_0), (x_0, y_{0.5})\}$. Then δ is a fuzzy topology on A and $\tau_\delta = \{\emptyset, S(A), \{x\}, \{y\}\}$ is a topology on $S(A)$ which is a T_1-space.
But (A,δ) is not a FT_0 -space. In fact $x_{0.3}\tilde{q}_A x_{0.1}$, but there is no $O_{x_{0.3}}$ such that $O_{x_{0.3}}\tilde{q}_A x_{0.1}$ and there is no $O_{x_{0.1}}$ such that $O_{x_{0.1}}\tilde{q}_A x_{0.3}$.

3.15 Theorem

Let (A,δ) be a fuzzy topological space. If (A,δ) is a FT_2 , then:
$x_\alpha = \cap \left\{\overline{(O_{x_\alpha})}_A :\ O_{x_\alpha} \in N_A(x_\alpha)\right\}$ for all $x_\alpha \in FP(A)$.
Proof. Let (A,δ) be a FT_2 space, $x_\alpha \in FP(A)$. Then for any $y_\beta \tilde{q} x_\alpha$ there exist $O_{y_\beta}, O_{x_\alpha} \in \delta$ such that $O_{y_\beta}\tilde{q}_A O_{x_\alpha} \Longrightarrow y_\beta \tilde{q}_A \bar{O}_{x_\alpha}$ for all O_{x_α} (by (iii) of Proposition (1.8)) and so, $y_\beta \tilde{q}_A \cap \bar{O}_{x_\alpha} \Longrightarrow x_\alpha \supseteq \cap \bar{O}_{x_\alpha}$ (by (6) of Proposition (1.2)). But clearly $x_\alpha \subseteq \cap \bar{O}_{x_\alpha}$.
Hence we get the result

3.16 Theorem

Let $(S(A),\tau)$ be a topological space. If $(S(A),\tau)$ is T_2-space, then (A,δ_τ) is FT_2 space.
Proof. Let (X,τ) be a T_2 -space , $x_\alpha \tilde{q}_A y_\beta$. Then either $x \neq y$ or ($x = y, \alpha+\beta \leq A(x)$).
If $x \neq y$, then there exist $O_x \in \tau$ and $O_y \in \tau$ such that $O_x \cap O_y = \phi$. Take $O_{x_\alpha} = \chi^A_{O_x} \in \delta_\tau$ and

$O_{y_\beta} = \chi^A_{O_y} \in \delta_\tau$, then $\chi^A_{O_x} \tilde{q}_A \chi^A_{O_y}$.
If $(x = y,\ \alpha + \beta \le A(x))$, then take $O_{x_\alpha} = \underline{\alpha} \cap A \in \delta_\tau$ and $O_{y_\beta} = \underline{\beta} \cap A \in \delta_\tau$ to be the required neighborhoods. Hence (A, δ_τ) is a FT_2 -space.

Note: The Example (3.4) in [7] shows that the converse of the above theorem may not be true in general, where consider X an infinite set and $A \in I^X$ be an infinite maximal subset of X.

The following example shows that, in general, a T_2-space$(S(A), \tau_\delta)$ need not imply that (A, δ) be a FT_2 space.

3.17 Example

Let $(S(A), \tau)$ be any T_2-space and $\delta = \Delta_\tau = \{\chi^A_B : B \in \tau\}$. Then $\tau_\delta = \tau$ which is a T_2-space. But (A, δ) is not a FT_2 space.

3.18 Theorem

Let (A, δ) be a fully stratified fuzzy topological space. If $(S(A), [\delta])$ is a T_2-space, then (A, δ) is a FT_2 space.

Proof. The proof is similar to that of Theorem (3.6).

3.19 Theorem

Let (A, δ) be a fully stratified and weakly induced fuzzy topological space. Then $(S(A), [\delta])$ is a T_2-space if and only if (A, δ) *is a* FT_2 -space.

Proof. Straightforward.

3.20 Definition[2]

A fuzzy topological space (A, δ) is said to be a fuzzy T_2-space if $\forall x_p, y_q \in_1 A$ with $(x \neq y), \exists\, U, V \in \delta$ such that $x_p \in_1 U, y_q \in_1 V$ and $U \tilde{q}_A V$, (where $x_p \in_1 A$ iff $p < A(x)$).

3.21 Theorem

Let (A, δ) be a fuzzy topological space. Then our FT_2 -space is a fuzzy T_2-space in the sense of the above definition.

Proof. Obvious and so is omitted.

The following example shows that the converse of the above theorem may not be true in general.

3.22 Example

Let $X = \{x, y, z\}$ and $A = (x_{0.6}, y_{0.5}, z_{0.4}) \in I^X$. Take $\delta = \{\underline{0}, A, (x_{0.6}, y_0, z_0), (x_0, y_{0.5}, z_0), (x_0, y_0, z_{0.4}), (x_{0.6}, y_{0.5}, z_0), (x_{0.6}, y_0, z_{0.4}), (x_0, y_{0.5}, z_{0.4})\}$. Then δ is a fuzzy topology on A which is a fuzzy T_2 space in the sense of Definition (3.20). But (A, δ) is not FT_2 space in our sense. In fact $x_{0.3} \tilde{q}_A x_{0.2}$ but there do not exist any two members of δ which not quasi-coincident referred to A members of δ containing $x_{0.3}$ *and* $x_{0.2}$.

6.3.23 Theorem

The following implications hold:

$FT_4 \Rightarrow FT_3 \Rightarrow FT_2 \Rightarrow FT_1 \Rightarrow FT_0$

Proof.

i) Let(A, δ)be a FT_4 space and $x_\alpha \tilde{q}_A B$, $B \in \delta'_A$. Then $\overline{x}_\alpha = x_\alpha \Longrightarrow \overline{x}_\alpha \tilde{q}_A B$. Since (A, δ) is a FR_3, then there exist $O_{\overline{x}_\alpha}, O_B \in \delta$ such that $O_{\overline{x}_\alpha} \tilde{q}_A O_B$. Now Put $O_{x_\alpha} = O_{\overline{x}_\alpha}$, then $O_{x_\alpha} \tilde{q}_A O_B$. Hence (A, δ) is a FT_3-space.

ii) Let(A, δ)be a FT_3 space and $x_\alpha \tilde{q}_A y_\beta$. Then $\overline{x}_\alpha = x_\alpha \Longrightarrow \overline{x}_\alpha \tilde{q} y_\beta$, $\overline{x}_\alpha \in \delta'_A$. Since (A, δ) is FR_2, then there exist$O_{\overline{x}_\alpha}$and O_{y_β} such that $O_{\overline{x}_\alpha} \tilde{q}_A O_{y_\beta}$. Now Put $O_x = O_{\overline{x}_\alpha}$, then $O_{x_\alpha} \tilde{q}_A O_{y_\beta}$. Hence (A, δ) is a FT_2-space.

iii) Let(A, δ) be a FT_2 space and let $x_\alpha \tilde{q}_A y_\beta$. Then there exist $O_{x_\alpha}$and O_{y_β} such that $O_{x_\alpha} \tilde{q}_A O_{y_\beta} \Longrightarrow y_\beta \tilde{q}_A O_{x_\alpha}$ and $x_\alpha \tilde{q}_A O_{y_\beta} \Longrightarrow x_\alpha \tilde{q}_A \overline{y}_\beta$and $\overline{x}_\alpha \tilde{q}_A y_\beta$ (by (iii) of Theorem (1.8)).
Hence (A, δ) is a FT_1-space.

iv) Obvious.

From the above theorem and Theorem (2.3), we obtain the following results.

3.24 Corollary

The following implications hold**:**

$$\begin{array}{ccccccccc} FT_4 & \Rightarrow & FT_3 & \Rightarrow & FT_2 & \Rightarrow & FT_1 & \Rightarrow & FT_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ FR_3 \wedge FR_0 & \Rightarrow & FR_2 & \Rightarrow & FR_1 & \Rightarrow & FR_0 & \Rightarrow & \text{any F-space.} \end{array}$$

4. Some Properties

4.1 Theorem

Let $(S(A),\tau)$ be an s-topological space on $(S(A)$. Then:
$(S(A),\tau)$ is a R_i-space $\Leftrightarrow$ $(A, w_A(\tau))$ is a FR_i-space, where $i = 0, 1, 2$.

Proof.

For $i = 0$. Suppose $(S(A),\tau)$ is R$_0$-space. Let $x_t \in FP(A)$ and let $O_{x_t} \in w_A(\tau)$. Then $x \in (O_{x_t})_\alpha \in \tau$ for any $\alpha < t$. Since $(S(A),\tau)$ is R$_0$, then $\overline{x} \subseteq (O_{x_t})_\alpha$ for any $\alpha < t$. It follows that $\overline{\alpha\chi^A_{\{x\}}} = \alpha\chi^A_{\overline{\{x\}}} \subseteq \alpha\chi^A_{(O_{x_t})_\alpha}$ for any $\alpha < t$. It is easy to see that $\overline{x}_t \subseteq \bigcup_{\alpha\le t}\overline{x}_\alpha = \bigvee_{\alpha\le t}\overline{\alpha\chi^A_{\{x\}}} \subseteq \bigvee_{\alpha\le t}\alpha\chi^A_{(O_{x_t})_\alpha} \subseteq \bigvee_{\alpha\in I}\alpha\chi^A_{(O_{x_t})_\alpha} = O_{x_t}$. Then $\overline{x}_t \subseteq O_{x_t}$ and so $(A, w_A(\tau))$ is FR$_0$.

Conversely, suppose $(A, w_A(\tau))$ is a FR$_0$-space , $x \in S(A)$ and let $O_{x_t} \in \tau$. Then $x_{A(x)} \subseteq \chi^A_{O_x} \in w_A(\tau)$. It follows that $\chi^A_{\overline{\{x\}}} = \overline{\chi^A_{\{x\}}} \subseteq \chi^A_{O_x}$ and so $\overline{x} \subseteq O_x$. Hence $(S(A),\tau)$ is a R$_0$-space.

For $i = 1$. Suppose $(S(A),\tau)$ is a R$_1$-space, $x_\alpha, y_\beta \in FP(A)$ *with* $x_\alpha \tilde{q}_A y_\beta$. Then either $x \neq y$ or $(x = y, \alpha + \beta \le A(x))$.

a) If $x \neq y$, then either $\overline{x} \neq \overline{y}$ or $\overline{x} = \overline{y}$ and so we have to cases:

i) If $\overline{x} \neq \overline{y}$, then there exist $O_x, O_y \in \tau$ such that $O_x \cap O_y = \phi$. Put $O_{x_\alpha} = \chi^A_{O_x} \in w_A(\tau)$ and $O_{y_\beta} = \chi^A_{O_y} \in w_A(\tau)$, then $\chi_{O_x}\tilde{q}_A O_{y_\beta}$. Hence $(A, w_A(\tau))$ is a FR$_1$-space.

ii) If $\overline{x} = \overline{y}$, this case is excluded (since $(S(A),\tau)$ is R_1).

b) If $x = y$, $\alpha + \beta \le A(x)$, then we put $O_{x_\alpha} = \underline{\alpha} \cap A$, $O_{y_\beta} = \underline{\beta} \cap A \in w_A(\tau)$ to be the required neighborhoods. Hence $(A, w_A(\tau))$ is a FR$_1$ space.

Conversely, suppose $(A, w_A(\tau))$ is a FR$_1$ and $x \notin \overline{y}$, then $x_{A(x)}\tilde{q}_A\overline{y}_{A(y)}$. Since $(A, w_A(\tau))$ is FR$_1$, then $\exists O_{x_{A(x)}}, O_{y_{A(y)}} \in w_A(\tau)$ such that $O_{x_{A(x)}}\tilde{q}_A O_{y_{A(y)}}$. Now put $O_x = (O_{x_{A(x)}})_{\frac{1}{2}A(x)} \in \tau$ and $O_x = (O_{y_{A(y)}})_{\frac{1}{2}A(y)} \in \tau$, then remains to prove that $O_x \cap O_y = \phi$.

Suppose $z \in (O_{x_{A(x)}})_{\frac{1}{2}A(x)} \cap (O_{y_{A(y)}})_{\frac{1}{2}A(y)} \Longrightarrow O_{x_{A(x)}}(z) > \frac{1}{2}A(x)$ and $O_{y_{A(y)}}(z) > \frac{1}{2}A(y) \Longrightarrow \frac{1}{2}A(y) > O'_{y_{A(y)}}(z)$. Now we have two cases. If $\frac{1}{2}A(x) > \frac{1}{2}A(y)$, then $O_{x_{A(x)}}(z) > O'_{y_{A(y)}}(z)$ which contradicts to $O_{x_{A(x)}} \subseteq O'_{y_{A(y)}}$. If $\frac{1}{2}A(y) > \frac{1}{2}A(x)$ we get the same contradiction.

Hence $(S(A),\tau)$ is a R$_1$-space.

For $i =$. Let $(S(A),\tau)$ be a R$_2$-space and $x_t \in FP(A)$ with $x_t \in U \in w_L(\tau)$. Now for any $t \le \alpha$ we have $x \in U_\alpha \in \tau$. Since $(S(A),\tau)$ is R$_2$ and so there exists $V \in \tau$ such that $x \in V \subseteq \overline{V} \subseteq U_\alpha$. It is clear that $x_t \in t\chi^A_V \subseteq t\chi^A_{\overline{V}} = \overline{t\chi^A_V} \subseteq t\chi^A_{U_\alpha} \subseteq U$.

Conversely, suppose $(A, w_A(\tau))$ is a FR$_2$-space $x \in U \in \tau$. Then we have $x_{A(x)} \in w_L(\tau)$ and so there exists $V \in w_L(\tau)$ such that $x_{A(x)} \in V \subseteq \overline{V}_A \subseteq \chi^A_U$. Then $x \in (x_{A(x)})_0 \in (V)_0 \subseteq (\overline{V}_A)_0 \subseteq \overline{(V)}_0 \subseteq U$. Hence $(S(A),\tau)$ is a R$_2$-space.

4.2 Theorem

Let $(S(A),\tau)$ be an s-topological space. Then $(S(A),\tau)$ is a T_i-space if and only if $(A, w_A(\tau))$ is a FT_i-space, for $i = 0, 1, 2, 3$.

Proof. For $i = 1$. Suppose $(S(A),\tau)$ is a T$_1$-space and $x_\alpha\tilde{q}_A y_\beta$. Then either $x \neq y$ or $(x = y,\ \alpha + \beta \le A(x))$. i) If $x \neq y$ then there exists $O_x \in \tau$ such that $y \notin O_x$ and there exists $O_y \in \tau$ such that $x \notin O_y$ and so $y_\beta\tilde{q}_A\chi^A_{O_x} = O_{x_\alpha}$(say) and $x_\alpha\tilde{q}_A\chi^A_{O_y} = O_{y_\beta}$(say), where $O_{x_\alpha}, O_{y_\beta} \in w_L(\tau)$.

ii) If $(x = y, \alpha + \beta \le A(x))$, then take $O_{x_\alpha} = \underline{\alpha} \cap A$, $O_{y_\beta} = \underline{\beta} \cap A \in w_A(\tau)$ to be the required neighborhoods. Hence $(A, w_A(\tau))$ *is a* FT_1-space.

Conversely, suppose $(A, w_A(\tau))$ is a FT$_1$-space and $x \neq y$. Then $x_\alpha\tilde{q}_A y_\beta$ $\forall \alpha, \beta$ belong to the range of A. In particular, we have $x_{A(x)}\tilde{q}_A y_{A(y)}$. Hence there exists $O_{x_{A(x)}} \in w_L(\tau)$ such that $O_{x_{A(x)}}\tilde{q}_A y_{A(y)}$ and there exists $O_{y_{A(y)}} \in w_A(\tau)$ such that $x_{A(x)}\tilde{q}_A O_{y_{A(y)}}$. Thus for any $0 < \alpha \le A(x)$ we have $y \notin (O_{x_{A(x)}})_\alpha \in \tau$ and $x \notin (O_{y_{A(y)}})_\alpha \in \tau$. Hence $(S(A),\tau)$ is a T$_1$-space.

The proof for the case $i = 0$ is similar to that of the case $i = 1$of the above theorem.

The proof for the case $i = 2$ is similar to the case $i = 1$ of the above theorem.

For $i = 3$. The proof follows from the cases $i = 2$ of Theorem(4.1) and from case $i = 1$ of the above theorem.

Now, in the following theorems we will show that the axioms (FR_i, for $i = 0, 1, 2$) and (FT_i, for $i = 0, 1, 2, 3$) are hereditary referred to the class of maximal subspaces.

4.3 Theorem

Let (A, δ) be a fuzzy topological space and Y be a maximal fuzzy subset of A. If (A, δ) is FR_i, then the fuzzy subspace (Y, δ_Y) is FR_i , for $i = 0, 1, 2$.

Proof.

As a sample we will prove the case $i = 2$ **.** Let (A, δ) be a FR_2-spasce , $x_\alpha \in FP(Y)$ and G be a fuzzy subset of (Y, δ_Y) with $x_\alpha \tilde{q}_Y G$. Then $x_\alpha \tilde{q}_Y \overline{G}^{\delta_Y}$. Since Y is a maximal fuzzy subset of A, then $\overline{G}^{\delta_Y} = Y \cap \overline{G}^{\delta}$ and so $x_\alpha \tilde{q}_Y \left(Y \cap \overline{G}^{\delta}\right) \Longrightarrow x_\alpha(z) + \min\left\{\overline{G}^{\delta}(z), Y(z)\right\} \leq Y(z)$.

Now if $Y(z) \neq 0 \Longrightarrow Y(z) = A(z)$, hence $x_\alpha(z) + \overline{G}^{\delta}(z) \leq A(z) \Longrightarrow x_\alpha \tilde{q}_A \overline{G}^{\delta}$. If $Y(z) = 0$, then $x_\alpha(z) = 0$ and so $x_\alpha(z) + \overline{G}^{\delta}(z) \leq A(z)$. So $x_\alpha \tilde{q}_A \overline{G}^{\delta}$. Since (A, δ) is FR_2 , then there exist O_{x_α} , $O_{\overline{G}^{\delta}} \in \delta$ such that $O_{x_\alpha} \tilde{q}_A O_{\overline{G}^{\delta}}$. Take $O^*_{x_\alpha} = O_{x_\alpha} \cap Y \in \delta_Y$ and $O_G = O_{\overline{G}^{\delta}} \cap Y \in \delta_Y$, then $O^*_{x_\alpha} \tilde{q}_Y O_G$ (since Y is a maximal fuzzy subset of A). Hence (Y, δ_Y) is a FR_2-space.

4.4 Theorem

Let (A, δ) be a fuzzy topological space and Y be a maximal closed fuzzy subset of A. If (A, δ) is FR_3, then the fuzzy subspace (Y, δ_Y) is FR_3.

Proof. Obvious and so is omitted.

4.5 Theorem

Let (A, δ) be a fuzzy topological space and Y be a maximal fuzzy subset of A. If (A, δ) is FT_i, then the subspace (Y, δ_Y) is FT_i , for $i = 0, 1, 2, 3$.

Proof.

As a sample we will prove the case $i = 0$ **.** Let (A, δ) be a FT_0, x_α , $y_\beta \in FP(Y)$ with $x_\alpha \tilde{q}_Y y_\beta$. Then $x_\alpha \tilde{q}_A y_\beta$. Since (A, δ) is FT_0, then there exists $O_{x_\alpha} \in \delta$ such that $O_{x_\alpha} \tilde{q}_A y_\beta$ or there exists $O_{y_\beta} \in \delta$ such that $O_{y_\beta} \tilde{q}_A x_\alpha$. Take $O^*_{x_\alpha} = O_{x_\alpha} \cap Y \in \delta_Y$ or $O^*_{y_\beta} = O_{y_\beta} \cap Y \in \delta_Y$. Since Y is a maximal fuzzy subset of A, then $O^*_{x_\alpha} \tilde{q}_Y y_\beta$ or $O^*_{y_\beta} \tilde{q}_Y x_\alpha$. Hence (Y, δ_Y) is a FT_0-space.

Remark

If Y any fuzzy subset of A, then the axioms (FR_i , for $i = 0, 1, 2$) and (FT_i , for $i = 0, 1, 2, 3$) need not be hereditary properties.

The following example shows the above remark.

4.6 Example

Let $X = \{x, y, z\}$ and $A = (x_{0.7}, y_{0.8}, z_{0.7}) \in I^X$. Take $\delta = \{\underline{0}, A, (x_{0.4}, y_{0.3}, z_{0.5}), (x_{0.3}, y_{0.3}, z_{0.2})\}$. Then δ is a fuzzy topology on A which is FR_3 and FR_0. Now let $Y = (x_0, y_0, z_{0.2})$ be any fuzzy subset of A. Then $\delta_Y = \{\phi, Y\}$ and so (Y, δ_Y) is a fuzzy subspace of (A, δ) but it is not a FR_0-space.

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