Logarithmic Finite Difference Method Applied To KdVB Equation.

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Abstract. A new finite difference scheme (Log FDM) improved for solving the linear or non linear higher order paritial differential equations. in this paper we will solve the (KdVB) equation. A comparison between The classical explicit finite difference method (FDM), exponential finite difference method (Exp FDM), the new suggested scheme (Log FDM) and the corresponding analytic solutions is done.

Keywords. Korteweg de Vries Burger equation, Nonlinear partial differentia equations, Finite Difference Method, Logarithmic Finite Difference Method.

1 Introduction

The Korteweg de Vries Burger equation have especial importance as it describe various physical phenomenas. The KdVB equation represent a mix of two important equations the Korteweg–de Vries equation (KdV) and Burger equation. The KdV describe the behavior of long waves in shallow water waves and waves of the plasma. It was discovered by Korteweg–de Vries in 1895. As it is an important equation many papers try to present it's analytic or numerical solutions. Adomian decomposition method (ADM) used in solving it in [1], Variational iteration method (VIM) [2], Homotopy perturbation method (HPM) [3] and many other analytical solution methods such as inverse scattering transform (IST) [4] and traveling wave solution [5]. The Burger's equation is a special case of the KdVB equation has been found to describe various kind of phenomena such as a mathematical model of turbulence [6] and the approximate theory of flow through a shock wave traveling in viscous fluid [7]. Fletcher using the Hopf–Cole transformation [8] gave an analytic solution of the system of two dimensional Burger's equations, Several numerical methods of this equation system have been given such as algorithms based on cubic spline function technique [9], applied an explicit–implicit method [10], implicit finite-difference scheme [11]. Soliman [12] used the similarity reductions for the partial differential equations to develop a scheme for solving the Burger's equation. As far as we know that little numerical works has been done to solve the KdVB equation. Recently a numerical method proposed for solving the KdVB equation by Zaki [13], he uses the collocation method with quintic B-spline finite element. The author [14] use the collocation solution of the KdV equation using septic splines as element shape function. Very recently Kaya [15] implement the Adomian decomposition method for solving the KdVB equation.

2 Derivation of KdV Burger equation and discussion

Consider an unmagnetized and collisionless plasma comprising of cold ions, isothermal positrons and superthermal electrons. We assume that the phase velocity of ion-acoustic wave is much smaller than the electron and positron thermal velocities and larger than the ion thermal velocity, so we therefore ignore the electron and positron inertia and write down the equation of motion for the ions. Such plasmas are described by the following normalized equations

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}=0 \tag{2.1}$$

$$\frac{\partial (\gamma u)}{\partial t}+u\frac{\partial (\gamma u)}{\partial x}+\frac{\partial \varphi}{\partial x}-\eta\frac{\partial^2 u}{\partial x^2}=0 \tag{2.2}$$

$$\frac{\partial^2 \varphi}{\partial x^2}=n_e-n-n_p \tag{2.3}$$

where n is the ion number density, u is the ion fluid velocity, η is the kinematic viscosity and φ is electrostatic potential. To model the effects of superthermal electrons, we have

$$n_e=\frac{1}{1-p}(1-\frac{\varphi}{k-1/2})^{-k-1/2} \tag{2.4}$$

The parameter k shapes predominantly the superthermal tail of the distribution and the normalization has been provided for any value of $k \geq 1/2$. In the limit $k \to \infty$, superthermal distribution reduces to the Maxwell–Boltzmann distribution. The positrons are assumed to be in thermal equilibrium, with the density

$$n_p = \frac{p}{1-p}\, e^{-\sigma\varphi} \tag{2.5}$$

In (2.1), the densities of the plasma species are normalized by the unperturbed ion density n_{io}, the ion velocity is normalized by the ion acoustic speed $c_i = \sqrt{T_e/m}$, space variables are normalized by the electron Debye length $\lambda_D = \sqrt{T_e/4\pi n_0 e^2}$, time variable is normalized by the electron plasma period $T = \sqrt{m_e/4\pi n_{e0} e^2}$ and electrostatic potential is normalized by the quantity (T_e/μ). The coefficient of kinematic viscosity η is incorporated in the parameter, $\eta = \frac{\eta}{\lambda_{Dcs}}$. Also, $p = n_{p0}/n_{e0}$ represents the positron concentration in e–p–i plasma and $\sigma = T_e/T_p$, is the temperature ratio of electron to positron. In order to investigate the propagation of ion acoustic shock waves and to derive the required KdVB equation in our e–p–i plasma, the independent variables are stretched as

$$\xi = \sqrt{\varepsilon}(x - ct), \qquad \tau = \sqrt{\varepsilon^3}t, \quad \eta = \sqrt{\varepsilon}\ \eta_0 \tag{2.6}$$

and the dependent variables are expanded as

$$\begin{aligned} n &= 1 + \varepsilon n_1 + \varepsilon^2 n_2 \\ u &= \varepsilon u_1 + \varepsilon^2 u_2 \\ \varphi &= \varepsilon\varphi_1 + \varepsilon^2\varphi_2 \end{aligned} \tag{2.7}$$

where ε is a small parameter which characterizes the strength of the nonlinearity, and λ is the phase velocity of the wave. Now, substituting (2.7) into (2.1), (2.2) and (2.3), using (2.6) and collecting the terms in the different powers of ε, we obtain the following equations at the lowest order of ε

$$n_1 = \frac{u_1}{\lambda}, \qquad u_1 = \frac{\varphi_1}{\lambda}\ , \qquad n_1 = \frac{1}{1-p}\left(p\sigma + \frac{2k+1}{2k-1}\right)\varphi_1 \tag{2.8}$$

and for the higher orders of ε, we have

$$\begin{aligned} &-\lambda\frac{\partial n_2}{\partial\xi} + \frac{\partial n_1}{\partial\tau} + \frac{\partial(n_1u_1)}{\partial\xi} + \frac{\partial u_2}{\partial\xi} = 0 \\ &-\lambda\frac{\partial u_2}{\partial\xi} + \frac{\partial u_1}{\partial\tau} + \frac{\partial\varphi_2}{\partial\xi} + u_1\frac{\partial u_1}{\partial\xi} - \eta_0\frac{\partial^2 u_1}{\partial\xi^2} = 0 \\ &\frac{\partial^2\varphi_1}{\partial\xi^2} = \frac{1}{1-p}\left(p\sigma + \frac{2k+1}{2k-1}\right)\varphi_2 - \frac{1}{4(1-p)}\left(\frac{(2k+1)(2k+3)}{(2k-1)^2} - p\sigma^2\right){\varphi_1}^2 - n_2 \end{aligned} \tag{2.9}$$

Finally the KdV–Berger equation is derived from (2.9)

$$\frac{\partial\varphi_1}{\partial\tau} + A\varphi_1\frac{\partial\varphi_1}{\partial\xi} + B\frac{\partial^3\varphi_1}{\partial\xi^3} - C\frac{\partial^2\varphi_1}{\partial\xi^2} = 0 \tag{2.10}$$

Where

$$A = \frac{1}{2}\left\{\frac{3}{\lambda} + \frac{\lambda^3}{(1-p)}\left(p\sigma^2 - \frac{(2k+1)(2k+3)}{(2k-1)^2}\right)\right\}, B = \frac{\lambda^3}{2},\ \ C = \frac{\eta_0}{2}. \tag{2.11}$$

3 Explanation of Logarithmic finite difference method (LFDM).

Consider the KdVB equation has the form [15]

$$u_t + \epsilon u u_x - v u_{xx} + \mu u_{xxx} = 0, \quad a \leq x \leq b \tag{3.1}$$

With the exact solution,

$$u(x,t) = \frac{6v^2}{25\epsilon\mu}\left[1 - \tanh\left(\frac{v}{10\mu}\left(x - \frac{6v^2}{25\mu}t\right)\right) + \frac{1}{2}sech^2\left(\frac{v}{10\mu}\left(x - \frac{6v^2}{25\mu}t\right)\right)\right] \tag{3.2}$$

where ϵ, μ and v are positive parameters. Eq.(3.1) is called the Korteweg-de Vries Burgers equation which derived by Su and Gardner [16], when the parameter $\mu = 0$, Eq.(3.1) will be the KdV equation and when the parameter $v = 0$, Eq.(3.1) will be Burger's equation. In our study, we will investigate three cases, the first one is the KdVB equation, the second one is the KdV in case of $v = 0$ and the third one is Burger's equation in case of $\mu = 0$.

In FDM the domain discretized to a finite number of points forming a mesh with horizontal step size $h = \frac{b-a}{N}$, N the number of intervals, $0 \le i \le N$ and the time step k. The derivatives replaced by a difference formulas as follows,

For $i = 1, 2$

$$(u_x)_i^n = \frac{-3u_i^j + 4u_{i+1}^j - u_{i+2}^j}{2h},$$

$$(u_{xx})_i^n = \frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{h^2} \tag{3.3}$$

$$(u_{xxx})_i^n = \frac{-5u_i^j + 18u_{i+1}^j - 24u_{i+2}^j + 14u_{i+3}^j - 3u_{i+4}^j}{2h^3}$$

For $i = 3 : N - 2$

$$(u_x)_i^n = \frac{u_{i+1}^j - u_{i-1}^j}{2h},$$

$$(u_{xx})_i^n = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{2h} \tag{3.4}$$

$$(u_{xxx})_i^n = \frac{u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j}{2h^3}$$

and For $i = N - 1, N$

$$(u_x)_i^n = \frac{3u_i^j - 4u_{i-1}^j + u_{i-2}^j}{2h},$$

$$(u_{xx})_i^n = \frac{2u_i^j - 5u_{i-1}^j + 4u_{i-2}^j - u_{i-3}^j}{h^2} \tag{3.5}$$

$$(u_{xxx})_i^n = \frac{5u_i^j - 18u_{i-1}^j + 24u_{i-2}^j - 14u_{i-3}^j + 3u_{i-4}^j}{2h^3}.$$

We assume that $F(u)$ be any continuous function multiply Eq.(3.1) by the derivative of F leads the following equation:

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = -\acute{F}(u)(\epsilon u u_x - v u_{xx} + \mu u_{xxx}) \tag{3.6}$$

$$\frac{\partial F}{\partial t} = -\acute{F}(u)(\epsilon u u_x - v u_{xx} + \mu u_{xxx}) \tag{3.7}$$

The usual difference formula for $\frac{\partial F}{\partial t}$ leads to,

$$F(u_i^{j+1}) = F(u_i^j) - k\acute{F}(u_i^j)\left(\epsilon u_i^j (u_x)_i^j - v(u_{xx})_i^j + \mu(u_{xxx})_i^j\right) \tag{3.8}$$

To obtain the Log FDM consider $F(u) = e^u$, then $\acute{F}(u) = e^u = F(u)$ Eq.(7) becomes,

$$F(u_i^{j+1}) = F(u_i^j)\left(1 - k\left(\epsilon(u_i^j)(u_x)_i^j - v(u_{xx})_i^j + \mu(u_{xxx})_i^j\right)\right) \tag{3.9}$$

Then

$$u_i^{j+1} = u_i^j + Ln\left(1 - k\left(\epsilon(u_i^j)(u_x)_i^j - v(u_{xx})_i^j + \mu(u_{xxx})_i^j\right)\right). \quad (3.10)$$

In Eq.(3.9) if we choose $F(u) = \ln(u)$, then $\acute{F}(u) = \frac{1}{u}$ which is the Exponential finite difference method (Exp FDM) that is developed by Bhattachary [17-18] and used to solve one dimensional heat conduction in a solid slab.

4 Numerical Experiments.

Case 1

For purpose of illustration of the LFDM for solving the KdVB equation, -10≤ $x \le 10$ in case of $\epsilon = 1$, v=2, $\mu = 1$ take $h = 1$ and $k = 0.0001$ for the KdVB equation, we start with an initial approximation,

$$u(x,0) = \frac{6v^2}{25\epsilon\mu}\left[1 - \tanh\left(\frac{v}{10\mu}x\right) + \frac{1}{2}sech^2\left(\frac{v}{10\mu}x\right)\right]$$

Table 1.1 the solution of KdVB equation at $t = .01$

x	FDM	EXP FDM	Log FDM	Exact
-10	1.9193820973	1.9193550075	1.9193820973	1.9193550075
-9	1.9186508551	1.9185900766	1.9186508551	1.9185900766
-8	1.9170781148	1.9169441558	1.9170781148	1.9169441558
-7	1.9137435172	1.9134583231	1.9137435171	1.9134583231
-6	1.9068234824	1.9062424733	1.9068234823	1.9062424733
-5	1.8929203876	1.8917806431	1.8929203874	1.8929203876
-4	1.8661903047	1.8640657581	1.8661903040	1.8640657581
-3	1.8177544627	1.8140506576	1.8177544607	1.8140506576
-2	1.7364516837	1.7305177966	1.7364516788	1.7305177966
-1	1.6122083588	1.6036178286	1.6122083486	1.6036178286
0	1.4418368221	1.4307400490	1.4418368052	1.4307400490
1	1.2339227517	1.2211948853	1.2339227296	1.2211948853
2	1.0081037419	0.9950816625	1.0081037189	0.9950816625
3	0.7879751376	0.7759397630	0.7879751179	0.7759397630
4	0.5925611879	0.5823385611	0.5925611736	0.5823385611
5	0.4318061237	0.4236810786	0.4318061146	0.4236810786
6	0.3071262349	0.3009836845	0.3071262296	0.3009836845
7	0.2145666808	0.2100895962	0.2145666780	0.2100895962
8	0.1479869849	0.1448078112	0.1479869834	0.1448078112
9	0.1011447626	0.0989325535	0.1011447619	0.0989325535
10	0.0686836668	0.0671786536	0.0686836665	0.0671786536

Case 2

Consider the KdV equation of the form, $u_t + \epsilon u u_x + \mu u_{xxx} = 0,\ 0 \le x \le 2,\ t > 0$. In this case $\epsilon = -6$, v=0, $\mu = 1$ take $h = 0.1$ and $k = 0.0001$ for the KdV equation, we start with an initial approximation, $u(x,0) = -2[sech^2(x)]$. The equation have the exact solution $u(x,t) = -2[sech^2(x - 4t)]$

Table 1.2 the solution of KdV equation at $t = .01$

x	FDM	EXP FDM	Log FDM	Exact
0.0	-1.999801243	-1.999801244	-1.999801245	-1.9968034102
0.1	-1.984675744	-1.984675787	-1.984675830	-1.9928172448
0.2	-1.930883275	-1.930883468	-1.930883648	-1.9496613041
0.3	-1.841695208	-1.841695556	-1.841695849	-1.8706675641
0.4	-1.724408429	-1.724408927	-1.724409287	-1.7616545412
0.5	-1.586474727	-1.586475311	-1.586475653	-1.6300551422
0.6	-1.436139102	-1.436139695	-1.436139954	-1.4839178553
0.7	-1.281149717	-1.281150256	-1.281150407	-1.3309915249
0.8	-1.128087401	-1.128087849	-1.128087906	-1.1780406596
0.9	-0.982018506	-0.982018853	-0.982018846	-1.0304505064

1.0	-0.846435144	-0.846435396	-0.846435357	-0.8921038894
1.1	-0.723399404	-0.723399579	-0.723399531	-0.7654646012
1.2	-0.613799344	-0.613799461	-0.613799416	-0.6517870265
1.3	-0.517640085	-0.517640160	-0.517640124	-0.5513797624
1.4	-0.434318776	-0.434318823	-0.434318796	-0.4638703825
1.5	-0.362856379	-0.362856408	-0.362856389	-0.3884396260
1.6	-0.302077304	-0.302077321	-0.302077309	-0.3240107763
1.7	-0.250738996	-0.250739007	-0.250738999	-0.2693919851
1.8	-0.207618962	-0.207618968	-0.207618963	-0.2233761592
1.9	-0.171563982	-0.171563986	-0.171563983	-0.1848060415
2.0	-0.141539172	-0.14153917	-0.141539172	0.15261267402

Case 3

Consider the Burger's equation of the form $u_t - \nu u_{xx} + \epsilon u u_x = 0,\ -10 \le x \le 10,\ t > 0$. with $\epsilon = 1$, ν=2, $\mu = 0$ take $h = 1$ and $k = 0.0001$for the Burger's equation, we start with an initial approximation, $u(x,0) = 2x$. The equation have the exact solution $u(x,t) = \frac{2x}{1+2t}$.

Table 1.3 the solution of Burger equation at $t = 0.01$.

x	FDM	EXP FDM	Log FDM	Exact
-10	-19.6077669804	-19.6078050637	-19.6085122058	-19.6078431372
-9	-17.6469902823	-17.6470245573	-17.6475940923	-17.6470588235
-8	-15.6862135843	-15.6862440509	-15.6866908180	-15.6862745098
-7	-13.7254368862	-13.7254635446	-13.7258023940	-13.7254901960
-6	-11.7646601882	-11.7646830382	-11.7649288306	-11.7647058823
-5	-9.80388349020	-9.80390253187	-9.80407013935	-9.80392156862
-4	-7.84310679216	-7.84312202549	-7.84322633116	-7.84313725490
-3	-5.88233009412	-5.88234151912	-5.88239741710	-5.88235294117
-2	-3.92155339608	-3.92156101274	-3.92158340825	-3.92156862745
-1	-1.96077669804	-1.96078050637	-1.96078431565	-1.96078431372
0	0.0	0.0	-0.00000001504	0.0
1	1.960776698040	1.960780506374	1.960769076417	1.960784313725
2	3.921553396081	3.921561012749	3.921523353708	3.921568627450
3	5.882330094122	5.882341519124	5.882262670352	5.882352941176
4	7.843106792162	7.843122025499	7.842987015222	7.843137254901
5	9.803883490203	9.803902531874	9.803696377181	9.803921568627
6	11.76466018824	11.76468303824	11.76439074507	11.76470588235
7	13.72543688628	13.72546354462	13.72507010774	13.72549019607
8	15.68621358432	15.68624405099	15.68573445403	15.68627450980
9	17.64699028236	17.64702455737	17.64638377222	17.64705882352
10	19.60776698040	19.60780506374	19.60701805203	19.60784313725

All computations are done by using Mathematica package at $t = 0.01$.

Conclusion

In this paper we have investigated The Log FDM, it is an effective method for solving linear or non linear partial differential equations for small times. In this work it used in solving KdVB, KdV and Burger equations. A comparison between Log FDM, Exp FDM, the classical FDM and the exact solution is done for every case. The numerical results show that the solution by Log FDM give a high accuracy very closed to the analytic solution and no more conditions or restrictions are needed.

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