Form of Optimal Control of Nonlinear Infinitely Space of Neutral Functional Differential Systems with Distributed Delays in the Control.

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Abstract. In this work, a Nonlinear Infinitely Space of Neutral Functional Differential Systems with Distributed Delays in the Control of the form:

$$\frac{d}{dt}(D(t,x_t)] = L(t,x_t)x_t + \int_{-\infty}^{0} A(t,\theta)x(t+\theta)d\theta + f(t,x_t)$$

$$+\int_{-h}^{0}[d_\theta H(t,\theta)]u(t+\theta) \qquad (1.1)$$

is presented for controllability analysis .We linearize the system (1.1) and obtain an expression for the solution of the system using the Unsymmetric Fubini Theorem as in the paper of **J.Klamka (1996)**. The set functions – reachable set, attainable set, target set upon which our studies hinged were extracted. We derived the form of the optimal control of the system 1.1) and expressed same using the definition of the signum function.

Key Words: Signum function, Reachable set, Attainable set, Controllability index, optimal control, Linearization Nonlinear System, Unsymmetric Fubini Theorem.

INTRODUCTION:

Systems with delay in the control, however pose the obvious challenge of how to handle the lags in the control and have provided multiple interest on the subject of controllability. They have diversified current thinking to accommodate the configuration of the complete state $z(t) = \{x(t), u_t\}$ *as it is transferred from the initial complete state to the final state with the pair* $(x(t), u_t)$ *changing values simultaneously to open up the area of study known as absolute controllability with zero in the interior of the reachable set. While the spontaneous interest in the transfer of the state system at* (x_0, u_{t_0}) *from initial time* t_0*, to the state* $x(t_1)$ *at time* t_1 *using any control gives impetus to the subject of relative controllability*

***Sebakby and M. N. Bayourni* (1973)** *blazed the trail by considering a finite set of first order differential equations of the form:*

$$\frac{d}{dt}\, x(t) = A(t)\, x(t) + B(t)\, U(t) + C(t) U(t-h)$$

Where, $A(t), B(t)$ *and* $C(t)$ *are nxn, nxm and nxm, matrices respectively and* $h > 0$ *is the delay time. They obtained a rank condition for controllability of the system which is rank* $[B, AB, A^2B, \dots, A^{n-1}B, C, AC, A^2C, \dots, A^{n-1}C] = n$.

This result has since been extended to systems with multiple delays (***see A. W. Olbrot* (1972)**).

The minimum energy control problem was addressed by ***J. Klamka* (1976)** ***and J. U. Onwuatu* (1990)** *and provided algebraic conditions for the relative controllability on the linear time varying system of the form:*

$$x(t) = A(t)x(t) + \int_{-h}^{0} [d_s\, H(t,s)]\, u(t+s).$$

Onwuatu in the same paper provided conditions for the relative null controllability of the system. He show that, if the system is relatively controllable on $[t_0, t_1]$*, with zero solution of the free part being uniformly asymptotically stable that the system would be relatively null controllable with constraint control.* ***Iheagwam V. A* (2005)**, *investigated the relative as well as absolute controllability of Ordinary Differential Systems with distributed delays in the control of the form:*

$$x(t) = A(t)x(t) + \int_{-h}^{0} [d_s\, H(t,s)]\, u(t+s) + f\big(t, x(t), u(t), u(t-h)\big).$$

Our principal objective is to investigate the form of the optimal control of the system of the form:

$$\frac{d}{dt}\, D(t, x_t) = L(t, x_t)x_t + \int_{-\infty}^{0} A(t)x(t+\theta)d\theta + f(t, xt) + \int_{-h}^{0} [d_\theta\, H(t,\theta)]\, u(t+\theta) \qquad (1.1)$$

and with zero in the interior of the reachable set using controllability standard or conditions to establish results.

2. *Notation and Preliminaries*:

Let n be a positive integer and $E = (-\infty, \infty)$ *be the real line.*
Denote by E^n *the space of real n − tuples called the Euclidean space with norm* $|.|$
if $J = [t_0, t_1]$ *is any interval of* E, L_2 *is the lebesgue space of square integrable functions from J to* E^n *written in full as* $L_2([t_0, t_1], E^n)$.

Let $h > 0$ *be a positive real number and let* $C([-h, 0], E^n)$ *be the Banach space of continuous function with the norm of uniform convergence, defined by*

$\|\phi\| = sup.\phi(s); \ -h \leq s \leq 0, for\ \phi \in C\ ([-h, 0], E^n),$

If x *is a function defined on* $[-h, 0]$ *to* $E^n (i.e.\ x : [-h, 0] \rightarrow E^n)$, *then* x_t *is a function defined on the delay interval* $[-h, 0]$ *given as*

$x_t(s) = x(t + s); s \in [-h, 0], \qquad t \in [0, \infty).$

Consider the nonlinear infinite neutral system (1.1). *That is*

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t)x_t + \int_0^{\infty} A(t)x(t + \theta)d\theta + f(t, x_t)$$
$$+ \int_{-h}^{0} [d_\theta H(t, \theta)]\, u(t + \theta) \qquad (2.1)$$

Where

$$L(t, x_t) = \sum_{k=1}^{\infty} Ak\ x(t - wk) + \int_{-\infty}^{0} A(t)x(t + \theta)d\theta$$

$$L(t, x_t)x_t = \int_{-h}^{0} d\theta\, \eta\, (t, s, x(t + s)x(t + \theta)$$

$\eta(t, s, \phi, \psi) \geq 0, for\ s \geq 0\ and\ \phi, \psi \in C = C\ ([-h, 0], E^n\ )$
$\eta(t, s, \phi, \psi)$ *is a continuous matrix function of bounded variation in* $s \in [-h, 0]$,
$var\ \eta(t) \leq m(t), m(t) \in L_1$, *where* L_1 *is the space of integrable functions.*
Let Ω *be an open subset of* $E\ x\ C$ *and* D *and* L *be bounded linear operators defined on* $E\ x\ C$ *into* E^n.
$D(t, x_t) = x(t)g(t, x_t), \qquad where$

$$g(t, x_t), \qquad = \sum_{n=0}^{\infty} A_n\,(t)\,\phi\,(t - w_n) + \int_{-h}^{0} A(t, s)\phi(s)ds = \int_{-h}^{0} d_\theta H(t, \theta)\,\phi(\theta)$$

$$Where\ 0 \leq w_n \leq h\ \ and\ \left| \int_{-h}^{0} d\theta\ H(t, \theta)\,\phi \right| \leq h(\theta)\|\phi\|$$

$\boldsymbol{D(t, x_t)}$ ***is non − atomic at zero*** (***differentiable and integrable at zero***).

$$\int_{-h}^{0} A(t, s)/ds + \sum_{n=1}^{\infty} |A_n(t)| \leq \delta(\varepsilon),\ for\ all\ t, where\ \delta(\varepsilon) \rightarrow 0.$$

f *is continuous and satisfies other smoothness conditions.*
Consider the system (2.1),

$$\frac{d}{dt}[D(t, x_t)] = L(t, x_t)x_t + \int_0^{\infty} A(t)x(t + \theta)d\theta$$
$$+ f(t, x_t) + \int_{-h}^{0} [d_\theta H(t, \theta)]\, u(t + \theta) \qquad (2.1)$$

(*circularity of the function from* $-\infty$ *to zero, and from zero to* ∞).

LINEARIZATION OF SYSTEM (2.1)

We can linearize the system (2.1) *as in* ***Chukwu*** (**1992**) *by setting* $x_t = z$, *a specified function inside the function without loss of generality.*
Thus the system (2.1) *becomes*

$$\frac{d}{dt}[D(t, x_t)] = L(t, z)x_t + \int_0^{\infty} A(t, \theta)x(t + \theta)d\theta$$

$$+\ f(t,x_t)\ +\int_{-h}^{0}[d_\theta\, H(t,\theta)]\, u(t+\theta) \qquad (2.2)$$

Evidently,

$$L(t,z)x_t\ =\ \sum_{k=0}^{\infty} A_k\, x(t-w_k)\ +\int_{-\infty}^{0} A(t,\theta)x(t+\theta)d\theta+\int_{0}^{\infty} A(t,\theta)x(t+\theta)d\theta \qquad (2.3)$$

$$L^*(t,z)x_t\ =\ \sum_{k=0}^{\infty} A_k\, x(t-w_k)+\int_{-\infty}^{\infty} A(t,\theta)x(t+\theta)d\theta \qquad (2.4)$$

The representations L, L^* are the same under the following assumptions

$$L(t,Z)x_t\ =\ \underset{p\to\infty}{\mathit{limit}}\sum_{k=0}^{P} A_k\, x(t-w_k)\ \ +\ \underset{M.N\to\infty}{\mathit{limit}}\int_{M}^{N} A(t,\theta)x(t+\theta)d\theta \qquad (2.5)$$

We assume the limits exist, giving finite partial sum for the infinite series and the improper integrals. Thus the system

$$L^*(t,z)x_t\ =\ \sum_{k=0}^{\infty} A_k\ x(t-w_k)+\int_{-\infty}^{\infty} A(t,\theta)x(t+\theta)d\theta \qquad (2.6)$$

is finite and well defined function.

In the light of the above, system (2.1) reduces to

$$\frac{d}{dt}\,[D(t,x_t)] = L(t,z)x_t\ + f\,(t,x_t)+\int_{-h}^{0}[d_\theta\, H(t,\theta)]\, u(t+\theta) \qquad (2.7)$$

$$x(to)=\phi\in\ C$$

Variation of Formula

Integrating system (2.7), after linearizing, we have

$$x(t)=X(t,t_0,\phi,u)x_0+\int_0^t X(t,s)+f(s,x_s)\,ds$$
$$+\int_0^t X(t,s)\left\{\int_{-h}^{0}[d_\theta\, H(t,\theta)]\, u(t+\theta)\right\}ds \qquad (2.8)$$

Where $X(t,s)$is the fundamental matrix of the homogeneous part of the system (2.7).
$X(t,s)\ =\ 1$ (identity matrix) for $t=s$, of nxn order.
The 3rd term in the right hand side of system (2.8)contains the values of the control $u(t)$ for $t<t_0$, as well as for $t>t_0$. $(t_0=0)$. The values of the control $u(t)$ for $t\in[t_0-h,t_0]$enter into the definition of initial complete state $\{x_0,u_{t_0}\}$.

To separate them, the 3rd term of the system (2.8)must be transformed by changing the order of integration.
Using the unsymmetric Bubini theorem, we have the following equalities;

$$x(t)=X(t,t_0,\phi,u)x_0+\int_0^t X(t,s)f(s,x_s)\,ds+\int_{-h}^{0} d_{H_\theta}\left(\int_0^t X(t,s)H(s,\theta)u(S+\theta)ds\right)$$

$$=X(t,t_0,\phi,u)x_0+\int_0^t X(t,s)f(s,x_s)\,ds+\int_{-h}^{0} d_{H_\theta}\left(\int_{0+\theta}^{t+\theta} X(t,s-\theta)H(s,-\theta,\theta)u(u-\theta+\theta)ds\right)$$

$$=X(t,t_0,\phi,u)x_0+\int_0^t X(t,s)f(s,x_s)\,ds+\int_{-h}^{0} d_{H_\theta}\left(\int_{0+\theta}^{t+\theta} X(t,s-\theta)H(s,-\theta,\theta)u(s)ds\right)$$

$$= X(t,t_0,\phi,u)x_0 + \int_0^t X(t,s)f(s,x_s)\,ds + \int_{-h}^0 d_{H_\theta}\left(\int_{0+\theta}^0 X(t,s-\theta)H(s,-\theta,\theta)u_0(s)ds\right)$$
$$+ \int_{-h}^0 d_{H_\theta}\left(\int_0^{t+\theta} X(t,s-\theta)H(s,-\theta,\theta)u(s)ds\right) \qquad (2.9)$$

Where the symbol d_{H_θ} *denotes that the integration is in the Lebesque − stielties sense with respect to the variable* θ *in the function* $H(t,\theta)$.
Let us introduce the following notations:

$$\widehat{H}(s,\theta) = \begin{cases} H(s,\theta) for\ s \le t, \theta \in R \\ 0, \quad for\ s > t, \ \ \theta \in R \end{cases} \quad \dots\dots\dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots\dots \qquad (2.10)$$

Hence $x(t)$ *can be expressed in the following form*:

$$x(t) = X(t,t_0,\phi,u)x_0 + \int_0^t X(t,s)f(s,x_s)\,ds + \int_{-h}^0 d_{H_\theta}\left(\int_{0+\theta}^0 X(t,s-\theta)H(s,-\theta,\theta)\,u_0(s)ds\right)$$
$$+ \int_{-h}^0 d_{H_\theta}\left(\int_0^t X(t,s-\theta)\widehat{H}(s,-\theta,\theta)u(s)ds\right) \ (2.11)$$

Using again the unsymmetric Fubini theorem, the equality (2.11) *can be rewritten in a more convenient form as follows* (***see J. Klamka*** (**1980**)).

$$x(t) = X(t,t_0,\phi,u)x_0 + \int_0^t X(t,s)f(s,x_s)\,ds + \int_{-h}^0 d_{H_\theta}\left(\int_{0+\theta}^0 X(t,s-\theta)H(s,-\theta,\theta)\,u_0(s)ds\right)$$
$$+ \int_0^t \left(\int_{-h}^0 X(t,s-\theta)\,d_{H_\theta}\widehat{H}(s-\theta,\theta)\right)u(s)ds \qquad (2.12)$$

Let us consider the solution $x(t)$ *of sytem* (2.1) *for* $t = t_1$.

$$x(t_1) = X(t,t_0,\phi,u)x_0 + \int_0^{t_1} X(t,s)f(s,x_s)\,ds + \int_{-h}^0 d_{H_\theta}\left(\int_{0+\theta}^0 X(t,s-\theta)H(s,-\theta,\theta)\,u_0(s)ds\right)$$
$$+ \int_0^{t_1} \left(\int_{-h}^0 X(t,s-\theta)d_{H_\theta}\,\widehat{H}(s-\theta,\theta)\right)u(s)ds$$

Definition 2.1: (***Complete state***)
The complete state for the system (2.1) *is given by the set* $z(t) = \{x_t\ , u_t\}$

Definition 2.2: (***Relative controllability***)
The system (2.1) *is said to be relatively controllable on* $[t_0,t_1]$ *if for every initial complete state* $z(0) = \{x_0, u_{t_0}\}$ *and* $x_1 \in E^n$, *there exists a control function* $u(t)$ *defined on* $[t_0,t_1]$ *such that the solution of system* (2.1) *satisfies* $x(t_1) = x_1$.

***Definition* 2.3: (*Reachable set*)**
The reachable set for the system (2.1) *is given as*

$$R(t_1,t_0) = \left\{\int_{t_0}^{t}\left(\int_{-h}^{0} X(t,s-\theta)d_{H_\theta}\,\widehat{H}(s-\theta,\theta)\right)u(s)d : u\epsilon U.\right\}$$

$Where\ U = \{u \in L_2([0,t],E^m: |u_j| \le 1,\ \ j = 1,2,3,\ldots..m\}$

***Definition* 2.4: (*Attainable set*)**
The attainable set for the system (2.1) *is given as*

$$A(t_1,t_0) = \{x(t,x_0,u): u \in U\ , and\ |u_j| \le 1, \quad for\ every\ j\},$$

$$where\ \ U = \{u \in L_2([0,t],E^m: |u_j| \le 1,\ \ j = 1,2,3,\ldots..m\}$$

***Definition* 2.5: (*Target set*)**
The target set for the system (2.1) *denoted by* $G(t_1,t_0)$ *is given as*

$$G(t_1,t_0) = \{x(\tau,x_0,u) :\ t_1 \ge \tau \ge t_0\ \ for\ fixed\ \ \tau\ and\ u \in U\}$$

***Definition* 2.6: (*Controllability Grammian*)**
The controllability grammian for the system (2.1) *is given as*

$$W(t_1,t_0) = \int_{t_0}^{t} Z(t,s)\, Z^T(t,s)ds$$
$$= \int_{t_0}^{t}\left[\int_{-h}^{0} X(t,s-\theta)d_{H_\theta}\,\widehat{H}(s-\theta,\theta)\right]\left[\int_{-h}^{0} X(t,s-\theta)d_{H_\theta}\,\widehat{H}(s-\theta,\theta)\right]^T ds$$

$Where,$ $$Z(t,s) = \left[\int_{-h}^{0} X(t,s-\theta)d_{H_\theta}\,\widehat{H}(s\ \theta,\theta)\right],$$ *and T denotes the matrix transpose.*

***Definition* 2.7 (*Properness*)**
The system (2.1) *is proper in* E^n *on* $[t_0,t_1]$, *if span* $R(t_1,t_0) = E^n$, *that is if*

$$C^T\left[\int_{-h}^{0} X(t,s-\theta)d_{H_\theta}\,\widehat{H}(s-\theta,\theta)\right] = 0\ \ a.e, \qquad t > 0 = t_0 \ \Rightarrow\ C = 0\,;\ C \in\ E^n.$$

***Definition* 2.8 (*Relative Controllability*)**
The system (2.1) *is relatively controllable on* $[t_0,t_1]$ *if*

$$A(t_1,t_0)\ \cap\ G(t_1,t_0) \ne \phi\,; t > o$$

2.1 *Controllability Standard or Conditions*

Application will be made of the following controllability conditions to establish results.

1. *The non emptiness of the intersection of two set functions – attainable set and target set is equivalent to the controllability. That is ;*

$$A(t_1,t_0)\ \cap\ G(t_1,t_0) \ne \phi, \Longrightarrow\ the\ system\ (2.1)\ is\ controllable.$$

2. *The controllability grammian or map* $W(t_1,t_0)$ *is invertible and the invertibility of the grammian guarantees; the controllability of the system. The invertibility of the grammian means that the rank of the grammian must be equal to n.*

It is also equivalent to $W(t_1, t_0)$ *is positive definite, which in turn is equivalent to*

$$C^T \left[\int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \widehat{H}(s-\theta, \theta) \right] = 0 \quad a.e, \qquad on\ [t_0, t_1]\ implies\ C = 0.$$

3. $0 \in interior\ R(t_1, t_0)$, *implies that the system* (2.1) *is controllable.*

3. *Main Results*

In this section, we shall derive the form of the optimal control of the system (1.1) *and express same using the definition of the signum function.*

***Definition* 3.1 (*Signum Function*)**

The signum function is defined by

$$\boldsymbol{sgn\ x} = \begin{Bmatrix} \mathbf{1}, \boldsymbol{if\ x} > 0 \\ -\mathbf{1}, \boldsymbol{if\ x} < 0 \end{Bmatrix}$$

Theorem 3.2 (Existence of an Optimal control)

Consider the system (1.1)

$$\frac{d}{dt}(D(t, x_t)] = L(t, x_t)x_t + \int_{-\infty}^{0} A(t, \theta)x(t+\theta)d\theta + f(t, x_t)$$

$$+ \int_{-h}^{0} [d_\theta H(t, \theta)]u(t+\theta) \qquad (1.1)$$

with its basic assumptions.

Suppose that the system (1.1) *is relatively controllable on the finite interval* $[t_0, t_1]$, *then there exists an optimal control* (***see Onwuatu*** (**1993**)).

Proof

By the relative controllability of the system (1.1), *the intersection conditions holds. That is*

$A(t_1, t_0) \cap G(t_1, t_0) \neq \phi$

Hence, $x(t, x_0, u) \in A(t_1, t_0)$ *and* $x(t, x_0, u) \in G(t_1, t_0)$.

Recall that the attainable set $A(t_1, t_0)$ *is translation of the reachable set* $R(t_1, t_0)$ *through the origin* ŋ, *where*

$$ŋ = X(t, t_0, \phi, u)x_0 + \int_{0}^{t} X(t, s) f(s, x_s)\, ds + \int_{-h}^{0} d_{H_\theta} \left(\int_{0+\theta}^{0} X(t, s-\theta) H(s, -\theta, \theta)\ uo(s) ds \right)$$

It follows that $y(t) \in R(t_1, t_0)$, $\quad for\ t \in [t_0, t_1]$, $\quad t_1 > t_0$, *and can be defined as*

$$y(t) = \int_{-h}^{0} d_{H\theta}\left(\int_{0}^{t} X(t, s-\theta)\widehat{H}(s, -\theta, \theta)u(s)ds\right)$$

$$= \int_{t_0}^{t}\left(\int_{-h}^{0} X(t, s-\theta)d_{H_\theta}\,\widehat{H}(s-\theta, \theta)\right)u(s)d$$

$Let,\ \ t^* = \ infimum\ \{ t : z(t) \in R\ (t_1, t_0)\}$

$Now\ \ t_0\ \le\ t_n\ \le\ t_1$ *and there is a sequence of times* t_n *and a corresponding sequence of control* $\{u_n\} \in U$ *with* $\{t_n\}$ *converging to* t^* *(the minimum time).*

$Let\ \ z(t_n)\ =\ y(t_n, u_n)\ \in R\ (t_1, t_0).$

Also ,

$$|z(t^*)\ -\ y(t^*, u_n)|\ = |z(t^*) - z(t_n)\ +\ z(t_n) - y(t^*, u_n)|$$

$$\le |z(t^*) - z(t_n)| + |\ y(t_n, u_n) - y\ (t^*, u_n)|$$

$$\le |z(t^*) - z(t_n)| + |y(t_n, u_n)\ - y(t^*, u_n)|$$

$$\le |z(t^*) - z(t_n)| + \int_{t^*}^{t_n} |y(s)|ds$$

By the continuity of $z(t)$ *which follows the continuity of* $R(t_1, t_0)$ *as a continuous set function and the integrability of* $\|y(t)\|$, *it follows that* $y(t^*, u_n) \to z(t^*)$ *as* $n \to \infty$ *, where* $z(t^*) = y\ (t^*, u^*) \in R(t_1, t_0)$

For same $u^* \in U$ *and by the definition of* t^* *,* u^* *is an optimal control.*

This establishes the existence of an optimal control for the Nonlinear Infinitely Space of Neutral Functional Differential Systems with Distributed Delays in the Control.

Theorem 3.2

Consider the Nonlinear InfinitelySpace of Neutral Differential Equation with Distributed Delays in the Control

$$\frac{d}{dt}(D(t, x_t)] = L(t, x_t)x_t + \int_{-\infty}^{0} A\,(t, \theta)x(t+\theta)d\theta + f(t, x_t)$$
$$+ \int_{-h}^{0} [d_\theta\, H(t, \theta)]u(t+\theta) \qquad (3.1)$$

with its basic assumptions, then u^* is the optimal control energy for the system (3.1) if and only if u^* is of the form:

$$\boldsymbol{u}^*(\boldsymbol{t}) = \boldsymbol{sgn}\left[\boldsymbol{C}^T \int_{\boldsymbol{t}_0}^{\boldsymbol{t}_1} [\int_{-\mathbf{h}}^{\mathbf{0}} \mathbf{X}\,(\mathbf{t}, \mathbf{s}-\boldsymbol{\theta})\mathbf{d}_{\mathbf{H}_{\boldsymbol{\theta}}}\widehat{\boldsymbol{H}}(\boldsymbol{s}-\boldsymbol{\theta}, \boldsymbol{\theta})\right], \qquad \boldsymbol{C} \in \boldsymbol{E}^{\boldsymbol{n}}$$

Proof

Suppose that, u^* *is the optimal control energy function for the system* (3.1), *then it maximizes the rate of change* $y(t, u)$ *, which is given as*:

$$y(t,u) = \int_{-h}^{0} X(t, s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) u(t),$$
$for\ u \in U, in\ the\ direction\ of\ C.$

$Since\ u(t)s\ are\ admissible\ controls, that\ is\ , they\ are\ constrained\ to\ lie\ in\ a\ unit\ sphere,$
$we\ have$

$$C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) u(t) \leq \left| C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right|$$

$$\leq C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) sgn \left[C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right] \ldots\ldots.\ldots (3.2)$$

$This\ inequality\ follows\ from\ the\ fact\ that,$ $\boldsymbol{for\ any\ non\ zero\ number\ \alpha, \alpha \leq sgn\ \alpha}.$
$Hence\ defining$

$$u^* = sgn[C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta)] \quad \ldots\ldots.(3.3)$$

$The\ inequality$ (3.2), $becomes$

$$C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) u(t) \leq C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) u^*(t)$$

$This\ shows\ that\ the\ control\ that\ maximizes\ y(t,u) \in R(t_1,t_0)\ is\ of\ the\ form$

$$u^* = sgn[C^T \int_{-h}^{o} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta)].$$

$Conversely,$

$$Let \quad , \quad u^* = C^T \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right],$$

$then\ for\ the\ control\ u \in U,$

$$C^T \int_{0}^{t} \left[\int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right] u(s) ds$$

$$\leq C^T \int_{t_o}^{t_1} \left[\int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right] sgn \left[C^T \int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right] ds.$$

$$\leq C^T \int_{t_o}^{t_1} \left[\int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right] ds, \qquad since\ \alpha \neq 0, sgn\alpha > 0.$$

$$\leq C^T \int_{t_o}^{t_1} [\int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) u^*(s)] ds$$

$This\ shows\ that\ u^*\ maximizes$ $\int_{0}^{t} \left[\int_{-h}^{0} X(t,s-\theta) d_{H_\theta} \hat{H}(s-\theta,\theta) \right]](s) ds$

$over\ all\ admissible\ controls\ u \in U.$
$Hence\ u^*\ is\ an\ optimal\ control\ for\ the\ system$ (3.1). $This\ completes\ the\ proof.$

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