A NUMERICAL APPROXIMATION FOR SOLVING THE FRACTIONAL-ORDER RICCATI EQUATION

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ABSTRACT. In this paper, is proposed a numerical approximation for solving fractional Riccati equation. The our numerical method is based on fractional-order Muntz-Legendre functions to abtain a numerical approximation of unknown functions. Also we use Caputo fractional derivative for fractional calculus. Finally, by applying the collocation method, The Riccati equation lead to a system of algebraic equations that easily solvable. Some numerical examples Show the efficiency and accuracy of the present method.

1. Introduction

It is well known that, fractional calculus is one of the important parts of the numerical analysis and physics. One of the most important branches that researchers pay attention to are fractional differential equations. Several articles and books have been published on these topics [1, 2, 3, 4]. Of these, some of equations are in standard models. For example population growth model [5, 6] and the nonlinear oscillation of earthquake can be modeled with fractional derivatives [7]. For a lot of them, an analytical solution is not available and they have to solve them in numerical methods. One of this kind of the fractional order differential equations is the Riccati equation. This equation is a nonlinear fractional differential equation of the from

$$a(t)D_*^{\alpha}u(t) + b(t)u(t) + c(t)u^2(t) = g(t), \tag{1}$$

$$u(0) = \beta, \qquad 0 < t < 1, \quad 0 < \alpha \leq 1, \tag{2}$$

where $a(t)$, $b(t)$, $c(t)$ and g(t) are known continuous functions on G = [0,1], and a is the order of fractional derivative in the Caputo sense. Several numerical methods applied to solve Eq. (1). For example Tau method [8], variational iteration method [9], and other methods [10, 11, 12, 13]. The aim of this paper

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is to apply the factional-order Muntz-Legendre functions (FMLFs) for solving fractional Riccati Eq. (1) (FRE), Using collection method.

This paper is organized as follows. Review of Caputo fractional derivative, is provided in Section 2. In Section 3, presents fractional-order Muntz-Legendre functions and its properties. Numerical method for solving Eq (1) is established in Section 4. Finally, we illustrate in Section 5, some numerical examples to show the efficiency and convergence of the proposed method.

2.Review of Caputo fractional derivative

3. 2.1. definition

The fractional derivative of y(t) in the Caputo sense is defined as

$$D_*^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau \tag{3}$$

for $0 < \alpha < 1$, $t > 0$.

2.2. properties

Some properties of the Caputo fractional derivative and operator D^a ,which will be used later, are as follows

i) $D_*^\alpha C = 0$, where C is a constant.

ii)

$$D_*^\alpha t^v = \begin{cases} \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & \alpha > 0,\ v > -1,\ t > 0,\ v \geq [\alpha], \\ 0, & v \in \mathbb{N}_0,\ v < [\alpha], \end{cases} \tag{4}$$

where [a] is the smallest integer greater than or equal to a and N_0 ={0,1, …}.

iii) the Caputo fractional derivative is a linear operation:

$$D_*^\alpha \Big(\sum_{i=1}^n a_i y_i(t) \Big) = \sum_{i=1}^n a_i D_*^\alpha y_i(t)$$

For more details about the properties of the Caputo fractional derivative see [2].

3. Review of fractional-order muntz polynomials

In this section, we give the definition and some properties of fractional-order Muntz-Legendre functions.

Definition 1. [15] The fractional-order Muntz-Legendre functions on the interval [0, T] are represented by the formula

$$L_n(t;\alpha) = \sum_{k=0}^n C_{n,k} \left(\frac{t}{T}\right)^{k\alpha}, \tag{5}$$

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where

$$C_{n,k} = \frac{(-1)^{n-k}}{\alpha^n k!(n-k)!} \prod_{v=0}^{n-1} ((k+v)\alpha + 1).$$

The function $L_k(t;\alpha)$, $k = 0, 1, \cdots, n$ form an orthogonal basis for $M_{n,\alpha} = \text{Span}\{1, t^\alpha, \cdots, t^{n\alpha}\}$, $t \in [0, T]$.
Also is satisfies

$$L_0(t;\alpha) = 1,\ L_1(t;\alpha) = (\frac{1}{\alpha} + 1)(\frac{t}{T})^\alpha - \frac{1}{\alpha},$$
$$b_{1,n} L_{n+1}(t;\alpha) = b_{2,n}(t) L_n(t;\alpha) - b_{3,n} L_{n-1}(t;\alpha),$$

where

$$b_{1,n} = a_{1,n}^{0,\frac{1}{\alpha}-1},\ b_{2,n}(t) = a_{2,n}^{0,\frac{1}{\alpha}-1}(2(\frac{t}{T})^\alpha - 1),\ b_{3,n} = a_{3,n}^{0,\frac{1}{\alpha}-1},$$
$$a_{1,n}^{\alpha,\beta} = 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta),$$
$$a_{2,n}^{\alpha,\beta}(x) = (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2],$$
$$a_{3,n}^{\alpha,\beta} = 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2).$$

Theorem 3.1. Let $L_n(t;\ a)$ *be a* fractional-order Muntz-Legendre functions, then we have the following Caputo fractional derivative of the functions $L_n(t;$ a): □

$$D_*^\alpha L_n(t;\alpha) = \sum_{k=1}^{n} D_{n,k} (\frac{t}{T})^{(k-1)\alpha} \tag{6}$$

where

$$D_{n,k} = \frac{T(1+k\alpha)}{T(1+k\alpha-\alpha)T^\alpha} C_{n,k}.$$

Proof. It is the result of Eqs. (4) and (5).

4.Functions approximation and description of the method Definition 2. An arbitrary function $u(t)$ can be expanded as follows:

$$u(t) \simeq u_m(t) = \sum_{i=0}^{m} a_i L_i(t;\alpha) = A^T \phi(t), \tag{7}$$

where

$$A = [a_0, a_1, \cdots, a_m]^T, \tag{8}$$

$$\phi(t) = [L_0(t;\alpha), L_1(t;\alpha), \ldots, L_m(t;\alpha)]^T. \tag{9}$$

Theorem 4.1. *Let $\phi(t)$ be a fractional Muntz-Legendre vector defined in* (9)*, then*

$$D_*^\alpha \phi(t) = [D_*^\alpha L_0(t;\alpha), D_*^\alpha L_1(t;\alpha), \ldots, D_*^\alpha L_m(t;\alpha)]^T, \tag{10}$$

where $D_*^\alpha L_i(t;\alpha)$ given in Eq. (6).

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Let $u_m(t)$ is an approximation for exact solution u(t) of Eq. (1), then all the terms of Eq. (1) can be expanded as

$$u(t) \simeq u_m(t) = \sum_{i=0}^{m} a_i L_i(t;\alpha) = A^T \phi(t), \tag{11}$$

$$D_*^\alpha u(t) \simeq D_*^\alpha u_m(t) = \sum_{i=0}^{m} a_i D_*^\alpha L_i(t;\alpha) = A^T D_*^\alpha \phi(t), \tag{12}$$

$$u^2(t) \simeq u_m^2(t) = \Big(\sum_{i=0}^{n} a_i L_i(t;\alpha)\Big)^2 = \Big(A^T \phi(t)\Big)^2. \tag{13}$$

(14)

For computation the Caputo fractional derivative in Eq. (12) using Eq.(10). Now, with suitably choice of collection points as $t\boldsymbol{j} = (jh)\ \ a, h = -\boldsymbol{T}., j = 0, \dots, m$ and employment collection method by substituting Eqs. (11) - (13) in Eqs. (1) and (2), we get a nonlinear system of algebraic equations as

$$F_0(A) = A^T \phi(0) - \beta = 0,$$
$$F_j(A) = a(t_j) A^T D_*^\alpha \phi(t) + b(t_j) A^T \phi(t_j) + c(t_j)\Big(A^T \phi(t_j)\Big)^2 - g(t_j) = 0, \quad j = 1, \cdots, m$$

After solving the nonlinear system

$$F(A) = \begin{bmatrix} F_0(A) \\ F_1(A) \\ \vdots \\ F_m(A) \end{bmatrix} = 0, \tag{15}$$

we can find A defined in Eq. (8) and approximate solution

$$u_m(t) = \sum_{k=0}^{m} a_k L_k(t;\alpha).$$

we can find A defined in Eq. (8) and approximate solution

$$E_m(\alpha) = \max\{|u(t_j) - u_m(t_j)|\}.$$

5-Numerical Illustration

In this section, we present some examples of FRE to show the efficiency and convergence of the proposed method. The results will be compared with the exact solutions and other methods. The accuracy of our method are estimated by maximum absolute error $E_m(a)$, which are given as follows:

$$Em(a) = \max\{|u(t_j) - u_m(t_j)|\}.$$

Example 1. Consider the FRE [14]

$$D_*^{\frac{1}{2}} u(t) + u(t) + u^2(t) = g(t), \quad t \in (0,1), \quad 0 < \alpha < 1,$$
$$u(0) = 0,$$

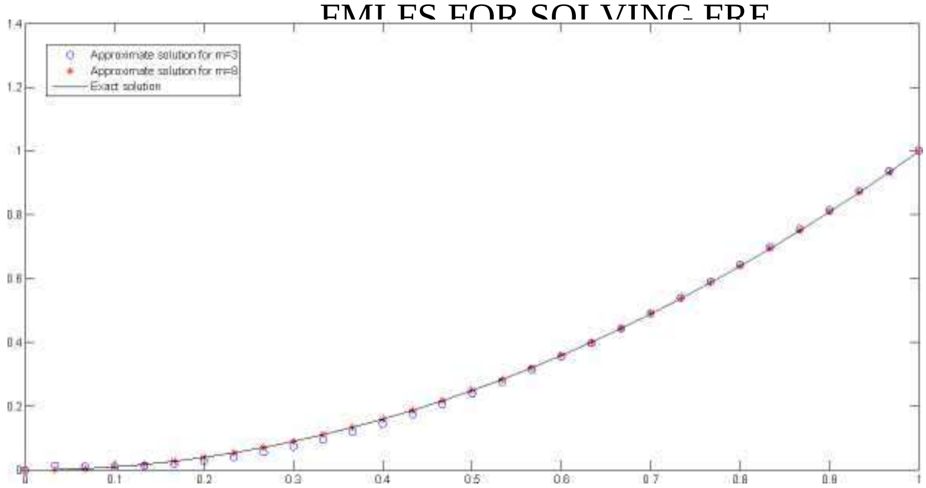

FIGURE 1. Numerical results for Example 1.

where

$$g(t) = \frac{8t^{\frac{3}{2}}}{3\sqrt{(\pi)}} + t^2 + t^4.$$

The exact solution is u(t) = t^2. The results are shown in Fig. 1. Fig. 1 shows comparison between the exact solution and approximate solution by the present method with m = 3 and m = 8. We see that proposed method has the high accuracy even with small m, (m = 8) in this example.

Example 2. *Consider the FRE* [14]

$$D_*^{\alpha} u(t) + u(t) + u^2(t) = g(t), \quad t \in (0,1), \quad 0 < \alpha < 1,$$
$$u(0) = 1,$$

where $g(t) = (E_\alpha(-t^\alpha))^2$ *and* $E_\alpha(z)$ *is the Mittag-Leffler function defined as*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

The exact solution is u(t) = $E_a(-t^a)$. The results are shown in Table 1 and Figs. 2 and 3. Table 1 shown a comparison of maximum absolute error $E_m(a)$ of present method with numerical method in [14]. Fig. 2 shows comparison between the exact solution and approximate solution with m =16 and various values of a. Fig. 3 shows the logarithmic graphs of $E_m(a)$ ($\log_{10} E_m(a)$). We see that the absolute error converges to zero as m ^ oo.

Example 3. *Consider the FRE* [14]

$$D_*^{\alpha} u(t) + tu(t) + 2u^2(t) = g(t), \quad t \in (0,1), \quad 0 < \alpha < 1,$$

TABLE 1. Maximum absolute error for Example 2.

a	E4 (a)	E8(a)	Ei2(a)	Ei6(a)	E25(a) in [14]
0.25	6.7145e - 05	5.3711e - 08	2.6507e - 11	3.5083e - 14	1.7e- 11
0.50	1.2287e - 04	2.3106e - 08	3.2019e - 10	1.1102e - 15	9. 3e - 12
0.75	1.0765e - 04	2.0881e - 09	9.3142e - 13	9.3259e - 14	7. 4e - 12
0.95	6. 5721e - 05	1.8347e - 10	4.5747e - 13	3.2752e - 13	3. 2e - 14

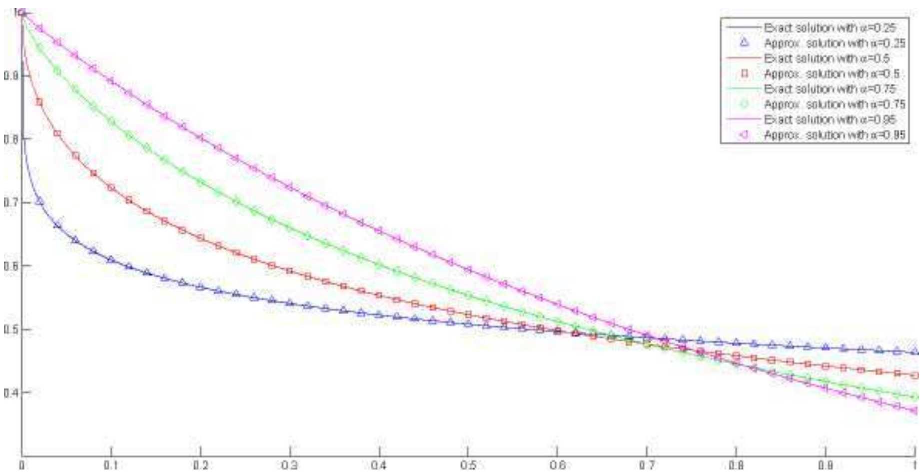

FIGURE 2. Comparison between the exact solution and approximate solution for Example 2.

$$u(0) = 0,$$

where

$$g(t) = -t^9 + 2t^{16} + 3t^{5+\frac{\alpha}{2}} - 12t^{12+\frac{\alpha}{2}} + \frac{63}{8}t^{1+\alpha} + 27t^{8+\alpha} - 27t^{4+3\frac{\alpha}{2}}$$
$$+ \frac{40320t^{8-\alpha}\Gamma(1-\alpha)}{\Gamma(9-\alpha)} - \frac{3t^{4-\frac{\alpha}{2}}\Gamma(1-\alpha)\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})} + \frac{9}{4}\Gamma(1-\alpha)\Gamma(1+\alpha).$$

The exact solution is $u(t) = t^8 - 3t^{4+\frac{\alpha}{2}} + \frac{9}{4}t^{\alpha}$.

The results are shown in Table 2 and Fig. 4. Table 2 shown a comparison of maximum absolute error $E_m(a)$ of present method with numerical method in [14]. We see that with increasing m, the absolute error converges to zero. Also, Fig. 4 shows comparison between the exact solution and approximate solution with m = 16 and various values of a.

6- Conclusion

In this paper, we applied fractional Muntz-Legendre functions (FMLFs) together with collocation method to obtian the numerical solution of nonlinear

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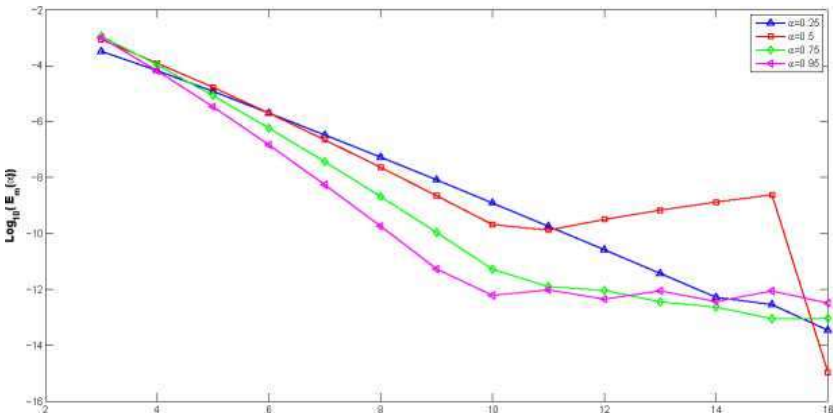

FIGURE 3. $E_m(a)$ convergence for Example 2.

TABLE 2. Maximum absolute error for Example 3.

a	E4 (a)	Eg (a)	Ei2(a)	$Eie(a)$	E25(a)in [14]
0.35	5.1925e - 02	4.9486e - 03	5.4553e - 05	1.4113e - 07	2. 1e - 06
0.50	7.0017e - 02	1.5892e - 03	2.3322e - 06	1.5152e - 12	4. 2e - 06
0.75	6.0276e - 02	1.9545e - 04	3.4748e - 09	8.8186e - 10	5. 1e - 07
0.90	7.7444e - 02	3.0473e - 05	7.4103e - 09	4.0646e - 10	4. 9e - 07
0.95	9.4292e - 02	6.5228e - 06	6.2094e - 08	3.8992e - 09	2. 2e - 07

fractional Riccati differential equation. We did a comparison between the exact solution and approximate solutionsome with our method and another method. Numerical examples show the efficiency and convergence of the our method.

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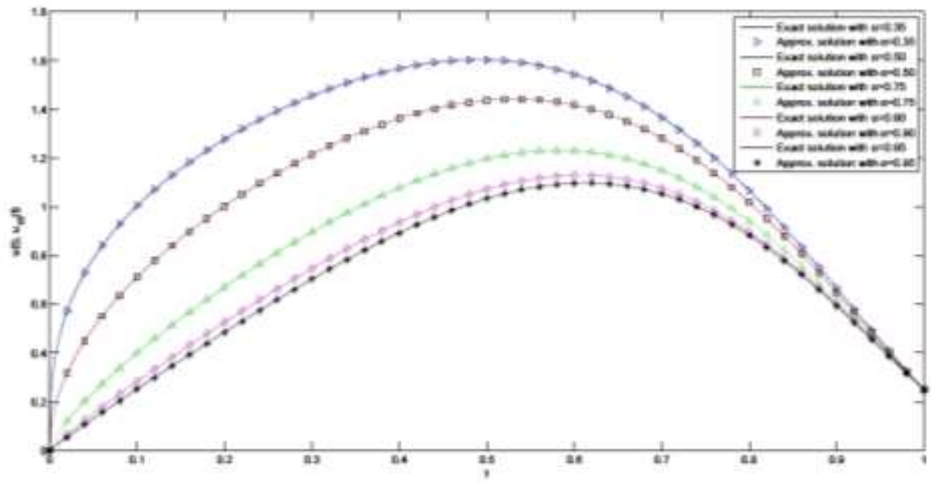

FIGURE 4. Comparison between the exact solution and approximate solution for Example 3.

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