Some properties of fuzzy closed sets and fuzzy semi-closed sets in fuzzy topological spaces defined on a fuzzy set

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Abstract. The aim of this work is to introduce certain types of fuzzy closed sets such as fuzzy regular closed set, fuzzy g – closed set, fuzzy g^* - closed, fuzzy rg – closed set and fuzzy sg – closed set. We studied the relation among them and some fundamental properties of these sets. Also we introduced and studied fuzzy regular open set, fuzzy g – open set, fuzzy g^* - open set, fuzzy rg – open set, also we gave the relation between them and some fundamental properties of those sets. Also we introduce the concept of a $\mathrm{f.s.g.c.s.}$ and study some properties of this fuzzy set. We depended a fuzzy topology which is defined on a fuzzy set instead of a crisp set X when we defined these sets.

1- Introduction

The concept of fuzzy set was introduced by Zedeh . The fuzzy topological space was introduced by [C. L. CHANG]. In 1970 Levine N. was introduced generalized closed set. T. Norini work we introduced certain types of fuzzy closed sets such as, fuzzy regular closed set, fuzzy closed set, fuzzy g - closed set, fuzzy g^* - closed set, fuzzy rg – closed set and fuzzy sg – closed set, also we introduced fuzzy regular open set, fuzzy open set, fuzzy g – open set, fuzzy g^* - open, fuzzy rg – open set, and we studied the relation between them and gave some fundamental properties of these sets. Also we introduce the concept of a $\mathrm{f.s.g.c.s.}$ and study some properties of this fuzzy set. We depend on a fuzzy topology space which defined in a fuzzy set instead of a crisp set X when we defined these fuzzy closed sets.

2- Fuzzy topology on a fuzzy set.

2-1 Definition [Zedeh L.A]: Let X be universe set. A fuzzy set $\tilde{A}$ in X is characterized by a membership function $M\tilde{A}: X \to [0,1]$, we can write a fuzzy set $\tilde{A}$ by

$\tilde{A} = \{(x, M\tilde{A}(x)) : x \in X, 0 \leq M\tilde{A}(x) \leq 1\}$.

The collection of all fuzzy sets in X will denoted by $I^x = \{\tilde{A},\ \tilde{A} \text{ is a fuzzy set in } X\}$.

2.2 Definition [C.K. Wong]: The support of a fuzzy set $\tilde{A}$ is the crisp set of all $x \in X$ such that $M\tilde{A}(x) > 0$ and denoted by $\mathcal{S}(\tilde{A})$

2.3 Definition [F. Conrad]: A fuzzy point P_x^r in X is a special fuzzy set with membership function defined by: $P_x^r = \begin{cases} r & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$

Where $0 < r \leq 1$, y is the support of P_x^r

2.4 Definition [C.K. Wong]: let $\tilde{A}, \tilde{B}$ be two fuzzy sets in X with membership function $M\tilde{A}(x)$ and $M\tilde{B}(x)$respectively, then for all $x \in X$:

1. $\tilde{A} \subseteq \tilde{B}$ if and only $M\tilde{A}(x) \leq M\tilde{B}(x)$
2. The complement of $\tilde{A}$ is denoted by $\tilde{A}^c$ and defined:
   $\tilde{A}^c = M\tilde{A}(x) = 1 - M\tilde{A}(x)$
3. $\tilde{A} = \tilde{B}$ if and only if $M\tilde{A}(x) = M\tilde{B}(x)$
4. $\tilde{A} \cap \tilde{B} = \min\{M\tilde{A}(x), M\tilde{B}(x)\}$
5. $\tilde{A} \cup \tilde{B} = \max\{M\tilde{A}(x), M\tilde{B}(x)\}$

2.5 Definition: A collection $\tilde{T}$ of fuzzy sets of $\tilde{A}$ such that $\tilde{T} \subseteq P(\tilde{A})$ is said to be a fuzzy topology on a fuzzy set $\tilde{A}$ if it satisfied the following conditions:

1. $\tilde{A}, \tilde{\phi} \in \tilde{T}$
2. The intersection of finite members of fuzzy sets of $\tilde{T}$ is an element of $\tilde{T}$.
3. The union of any members of $\tilde{T}$ is an element of $\tilde{T}$.
   $(\tilde{A}, \tilde{T})$ is called a fuzzy topological space and denoted by a $\mathrm{f.t.s.}$

2.6 Remark:

1. The members of $\tilde{T}$ are called fuzzy open set, and denoted by $\mathrm{f.o.s.}$
2. If $\tilde{B} \in \tilde{T}$, then the complement of $\tilde{B}$ is called a fuzzy closed set $(\mathrm{f.c.s})$ and defined as $M\tilde{B}(x) = M\tilde{A}(x) - M\tilde{B}(x)$

Where $\tilde{A}$ as a universe fuzzy set on which $\tilde{T}$ is defined.

3- Certain types of fuzzy closed sets.

3.1 Definition: Let $(\tilde{A}, \tilde{T})$ be a $\mathrm{f.t.s}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A}, \tilde{T})$, then $\tilde{B}$ is called a fuzzy regular closed set briefly $(\mathrm{f.r.c.s.})$ if

$M\tilde{B}(x) = M\overline{\tilde{B}^{\circ}}(x)$

3.2 Example: Let $\tilde{A} = \{(a, 0.8), (b, 0.8), (c, 0.8)\}$ be a universe fuzzy set. Let $\tilde{B} = \{(a, 0.8), (b, 0.0), (c, 0.8)\}$ and let $\tilde{B} = \{(b, 0.8), (b, 0.8), (c, 0.8)\}$ are fuzzy sets on $\tilde{A}$, then $\tilde{T} = \{\tilde{\phi}, , \tilde{A}, \tilde{B}, \tilde{C}\}$ is a fuzzy topology on $(\tilde{A}, \tilde{T})$.

Now take $\tilde{B} = \{(a, 0.8), (b, 0.0), (c, 0.8)\}$ then $\tilde{D}^{\circ} = \{(a, 0.8), (b, 0.0), (c, 0.8)\}$, and $\overline{\tilde{D}}^{\circ} = \{(a, 0.8), (b, 0.0), (c, 0.8)\}$.

It is clear that $M\tilde{B}(x) = M\overline{\tilde{B}}^{\circ}$, therefore $\tilde{B}$ is a $\mathrm{f.r.c.s.}$ in $(\tilde{A}, \tilde{T})$.

3.3 Proposition: Every $\mathrm{f.r.c.s.}$ is a fuzzy closed set.

Proof: It is clear.

3.4 Remark: The converse of proposition need not be true as the following example:

3.5 Example: let $\tilde{A} = \{(a, 0.7), (b, 0.6), (c, 0.3)\}$ be a universe fuzzy set. Let $\tilde{B} = \{(a, 0.6), (b, 0.4), (b, 0.2)\}$ be a fuzzy set. Then $\tilde{T} = \{\tilde{\phi}, \tilde{A}, \tilde{B}\}$ be a fuzzy topology on $\tilde{A}$. Now let $\tilde{D} = \{(a, 0.1), (b, 0.2), (c, 0.1)\}$ then $\tilde{D}^{\circ} = \tilde{\phi}$ and $\bar{\tilde{D}}^{\circ} = \tilde{\phi}$, therefore $M\tilde{D}(x) \neq M\bar{\tilde{D}}^{\circ}$ and hence $\tilde{D}$ is not $\mathrm{f.r.c.s.}$

3.6 Definition: Let $(\tilde{A}, \tilde{T})$ be a $\mathrm{f.t.s.}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A}, \tilde{T})$, then $\tilde{B}$ is called a fuzzy regular open set briefly $(\mathrm{f.r.o.s.})$ if

$M\tilde{B}(x) = M\bar{\tilde{B}}^{\circ}$

3.7 Proposition: Every $\mathrm{f.r.o.s.}$ is a fuzzy open set.

Proof: It is clear.

3.8 Remark: the converse of proposition (3.7) need not be true as the following:

3.9 Example: In the example (3.5), we see that $\tilde{B} = \{(a, 0.6), (b, 0.4), (c, 0.2)\}$ is a fuzzy open set in $(\tilde{A}, \tilde{T})$, and $M\bar{\tilde{B}}(x) = M\tilde{A}(x)$ also $M\bar{\tilde{B}}^{\circ} = M\tilde{A}(x)$

Therefore $M\tilde{B}(x) \neq M\bar{\tilde{B}}^{\circ}$, so $\tilde{B}$ is not $\mathrm{f.r.o.s.}$

3.10 Proposition: Let $(\tilde{A}, \tilde{T})$ be f.t.s. and let $\tilde{B}$ be a fuzzy open set in $(\tilde{A}, \tilde{B})$, then $\bar{\tilde{B}}^{\circ}$is a $\mathrm{f.r.o.s.}$ in $(\tilde{A}, \tilde{T})$.

Proof: Let $\tilde{B}$ be a fuzzy open set in $(\tilde{A}, \tilde{T})$.

First we are going to show that

$M\bar{\tilde{B}}^{\circ} \leq M\left(\overline{\left(\bar{\tilde{B}}^{\circ}(x)\right)}\right)^{\circ}$

Now since $M\tilde{B}(x) \leq M\overline{\tilde{B}(x)}$, then $M\bar{\tilde{B}}^{\circ} \leq M\bar{\tilde{B}}^{\circ}(x)$but $\tilde{B}$ is a fuzzy open set in $(\tilde{A}, \tilde{T})$. (By hypothesis), then $M\tilde{B}^{c}(x) = M$, and then

$M\tilde{B}(x) \leq M\bar{\tilde{B}}^{\circ}(x)$.

Therefore $\bar{\tilde{B}}(x) \leq M\overline{\left((\bar{\tilde{B}})^{\circ}\right)}$. And thus

$M(\overline{\tilde{B}(x)})^{\circ} \leq M\overline{\left(\bar{\tilde{B}}^{\circ}(x)\right)}^{\circ}$

Second we are going to show that

$M\overline{(\bar{\tilde{B}}^{\circ})}^{\circ} \leq M\bar{\tilde{B}}^{\circ}(x)$

Since $M\bar{\tilde{B}}^{\circ}(x) \leq M\bar{\tilde{B}}(x)$

Then $M\overline{\bar{\tilde{B}}(x)}^{\circ} \leq M\bar{\bar{\tilde{B}}}(x)$

But $M\bar{\bar{\tilde{B}}}(x) = M\bar{\tilde{B}}(x)$

And so $M\overline{\bar{\tilde{B}}^{\circ}} \leq M\bar{\tilde{B}}(x)$

Then $M\overline{\left(\bar{\tilde{B}}^{\circ}(x)\right)}^{\circ} \leq M\bar{\tilde{B}}^{\circ}(x)$

Therefore $M\bar{\tilde{B}}^{\circ}(x) = M\overline{\left(\bar{\tilde{B}}^{\circ}\right)}^{\circ}$

Thus $\bar{\tilde{B}}^{\circ}$ is a f.r.o.s. in $(\tilde{A},\tilde{T})$.

<u>3.11 Proposition</u>: Let $(\tilde{A},\tilde{T})$ be a f.t.s. and let $\tilde{B}$ and $\tilde{C}$ be a disjoint fuzzy open set in $(\tilde{A},\tilde{T})$ then:

$\min\{M\bar{\tilde{B}}^{\circ}(x), M\overline{C\ (x)}^{\circ}\} = 0 \ \forall\, x \in X$

<u>Proof</u>: Let $\tilde{B}$ and $\tilde{C}$ be a fuzzy open sets in $(\tilde{A},\tilde{T})$ such that

$\min\{M\bar{\tilde{B}}^{\circ}(x), M\overline{C\ (x)}^{\circ}\} = 0 \ \ \forall\, x \in X$

Then $M\tilde{B}(x) \leq M\tilde{C}^{c}(x)\ \forall\, x \in X$ and $M\tilde{C}^{c}(x) \leq M\tilde{B}^{c}(x)\ \forall\, x \in X$

And then $M\bar{\tilde{B}}(x) \leq MM\tilde{C}^{c}(x)$ and $M\bar{\tilde{C}}(x) \leq M\tilde{B}(x)\ \forall\, x \in X$

Since $M\overline{\tilde{B}(x)} \leq M\bar{\tilde{B}}(x)$ and $M\overline{\tilde{C}(x)}^{\circ} \leq M\bar{\tilde{C}}(x)$

Then $M\bar{\tilde{B}}^{\circ}(x) \leq M\bar{\tilde{B}}(x) \leq M\tilde{C}^{c}(x)$ and

$M\bar{\tilde{C}}^{\circ}(x) \leq M\bar{\tilde{C}}(x) \leq M\tilde{B}^{c}$

So $\min\{M\bar{\tilde{B}}^{\circ}(x), \tilde{C}(x)\} = 0$ and $\min\{M\bar{\tilde{C}}^{\circ}(x)\ , M\tilde{B}(x)\} =$

Therefore $\min\{\bar{\tilde{B}}^{\circ}(x), M\bar{\tilde{C}}^{\circ}\} = 0$

<u>3.12 Definition</u>: Let $(\tilde{A},\tilde{T})$ be a f.t.s. and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is said to be a fuzzy generalized closed set briefly (f.g.c.s) if $M\overline{\tilde{B}(x)} \leq M\tilde{O}(x)$ whenever $M\tilde{B}(x) \leq M\tilde{O}(x)$, where $\tilde{O}$ is a fuzzy open set in $(\tilde{A},\tilde{T})$.

<u>3.13 Example</u>: Let $\tilde{A} = \{(a, 0.8), (\ b, 0.8), (c, 0.8)\}$ be a universe fuzzy set.

And let $\tilde{B} = \{(a, 0.8), (b, 0.0), (c, 0.0)\}$, $\tilde{C} = \{(a, 0.0), (b, 0.8), (c, 0.8)\}$ be a fuzzy sets of $\tilde{A}$.

Then $\tilde{T} = \{\tilde{\phi}, \tilde{A}, \tilde{B}, \tilde{C}\}$ be a f.t.s. defined on $\tilde{A}$.

Take $\tilde{D} = \{(a, 0.0), (b, 0.8), (c, 0.0)\}$ be a fuzzy set in $(\tilde{A},\tilde{T})$.

We notice that the fuzzy open sets which contain $\tilde{D}$ are $\tilde{C}$ and $\tilde{A}$ and which contain $\bar{\tilde{D}}$ too, therefore $\tilde{D}$ is a f.g.c.s. in $(\tilde{A},\tilde{T})$.

<u>3.14 Proposition</u>: Every fuzzy closed set is a f.g.c.s.

<u>Proof</u>: It is clear.

3.15 Remark: The converse of proposition (3.14) need not be true. In the example (3.13) it is clear that $\tilde{B}$ is a f.g.c.s. but is not fuzzy closed set.

3.16 Corollary: Every $\mathrm{f.r.c.s.}$ is a $\mathrm{f.g.c.s.}$

3.17 Definition: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s.}$ and let be a fuzzy set of $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is said to be fuzzy generalized open set briefly ($\mathrm{f}.g.\mathrm{o.s}$) if $\tilde{B}^c$ is a $\mathrm{f}.g.\mathrm{c.s.}$

3.18 Proposition: Every fuzzy open set is a $\mathrm{f}.g.\mathrm{o.s.}$

3.19 Remark: The converse of the proposition (3.18) need not be clear as the following example:

3.20 Example: In the example (3.13) take $\tilde{E}=\{(a,0.8),(b,0.0),(c,0.8)\}$ since $\tilde{B}^{\circ}$ is a $\mathrm{f}.g.\mathrm{c.s.}$ then $\tilde{E}$ is a $\mathrm{f}.g.\mathrm{o.s.}$ and it is clear that $\tilde{E}$ is not fuzzy open set.

3.21 Proposition: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s.}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then $M\left(\tilde{B}^{\circ}(x)\right)^c = M\overline{\left(\tilde{B}^c(x)\right)}$

3.22 Theorem: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s.}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is a $\mathrm{f}.g.\mathrm{o.s.}$ if only if

$M\tilde{G}(x)\le M\tilde{B}^{\circ}(x)$ whenever $M\tilde{G}(x)\le M\tilde{B}$ where $\tilde{G}$ is a fuzzy closed set in $(\tilde{A},\tilde{T})$.

Proof: depending on proposition (3.21)

3.23 Definition: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s.}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$ then $\tilde{B}$ is said to be a fuzzy g^*- closed set briefly ($\mathrm{f}.g^*.\mathrm{c.s.}$) if $M\bar{\tilde{B}}(x)\le M\tilde{u}(x)$ whenever $M\tilde{B}(x)\le M\tilde{u}(x)$ where $\tilde{u}$ is a $\mathrm{f}.g.\mathrm{o.s.}$ in $(\tilde{A},\tilde{T})$. And a fuzzy set $\tilde{C}$ of $(\tilde{A},\tilde{T})$ is said to be a fuzzy g^* - open set briefly $\mathrm{f}.g^*.\mathrm{o.s}$ if $\tilde{C}^c$ is $\mathrm{f}.g^*.\mathrm{c.s.}$

3.24 Example: Let $\tilde{A}=\{(a,0.9),(b,0.8),(c,0.7)\}$ be a universe fuzzy set. And let $\tilde{B}=\{(a,0.7),(b,0.4),(c,0.6)\}$ be a fuzzy set of $\tilde{A}$ then $\tilde{T}=\{\tilde{\phi},\tilde{A},\tilde{A}\}$ be a $\mathrm{f.t.s.}$ on $\tilde{A}$.
Take $\tilde{D}=\{(a,0.8),(b,0.4),(c,0.6)\}$

Then $M\bar{\tilde{D}}(x)=M\tilde{A}(x)$ and $\tilde{A}(x)$ is the only $\mathrm{f}.g.\mathrm{o.s.}$ which contains $\tilde{D}$ and $\tilde{A}(x)$ contains $\bar{\tilde{D}}$ too, therefore $\tilde{D}$ is a $\mathrm{f}.g^*.\mathrm{c.s.}$ in $(\tilde{A},T)$.

3.25 Proposition: Every a $\mathrm{f}.g^*.\mathrm{c.s.}$ is a $\mathrm{f}.g.\mathrm{c.s.}$

3.26 Remark: The converse of proposition (3.25) need not be true as the following example:

3.27 Example: In the example (3.13) it is clear that

$\widetilde{D} = \{(a, 0.0), (b, 0.8), (c, 0.0)\}$ is a f. g. c. we are going to explain that $\widetilde{D}$ is not f. g^*. c. s. we notice that $\widetilde{D}$ is a f. g. o. s. in $(\tilde{A}, \tilde{T})$, then $M\bar{\bar{\widetilde{D}}}(x) \leq M\bar{\widetilde{D}}(x)$ but $M\bar{\widetilde{D}}(x) = M\tilde{A}(x) \nleq M\widetilde{D}(x)$. Therefore $\widetilde{D}$ is not f. g^*. c. s.

3.28 Proposition: Every fuzzy closed set is a f. g^*. c. s.

3.29 Remark: The converse of proposition (3.28) need not be true as the following example:

3.30 Example: In example (3.24) it is clear that $\widetilde{D}$ is a f. g^*. c. s. but not fuzzy closed set.

3.31 Proposition: Let $(\tilde{A}, \tilde{T})$ be a f. t. s. and let $\tilde{B}$ be a f. g. o. s. and f. g^*. c. s. in $(\tilde{A}, \tilde{T})$ then $\tilde{B}$ is a fuzzy closed set.

Proof: Let $\tilde{B}$ be a f. g. o. s. and f. g^*. c. s. in $(\tilde{A}, \tilde{T})$.

We are going to show that $\tilde{B}$ is a fuzzy closed set in $(\tilde{A}, \tilde{T})$.

Now $M\tilde{B}(x) \leq M\tilde{B}(x)$ and $M\bar{\bar{\tilde{B}}}(x) \leq M\tilde{B}(x)$

But $M\tilde{B}(x) \leq M\bar{\bar{\tilde{B}}}(x)$, so $M\bar{\bar{\tilde{B}}}(x) = M\tilde{B}(x)$

Therefore $\tilde{B}$ is a fuzzy closed set in $(\tilde{A}, \tilde{T})$.

3.32 Definition: Let $(\tilde{A}, \tilde{T})$ be a f. t. s. and let $\tilde{B}$ be a fuzzy set in $(\tilde{A}, \tilde{T})$ then $\tilde{B}$ is said to be a fuzzy regular generalized closed set briefly f.r.g.c.s. if $M\bar{\bar{\tilde{B}}}(x) \leq M\tilde{u}(x)$ whenever $M\tilde{B}(x) \leq M\tilde{u}(x)$ where $\tilde{u}$ is f. r. o. s. in $(\tilde{A}, \tilde{T})$.

3.33 Proposition: Every f. g. c. s. is a f. r. g. c. s.

Proof: Let $(\tilde{A}, \tilde{T})$ be a f. t. s. and let $\tilde{B}$ be a f. g. c. s. in $(\tilde{A}, \tilde{T})$, let $M\tilde{B}(x) \leq M\tilde{u}(x)$ where $\tilde{u}$ be f. r. o. s. in $(\tilde{A}, \tilde{T})$

Then $\tilde{u}$ is a fuzzy open set (Proposition 3.7) now since $\tilde{B}$ is a f. g. c. s. and $\widetilde{U}$ is a fuzzy open set, then $M\bar{\tilde{B}}(x) \leq M\tilde{u}(x)$

Therefore $\tilde{B}$ is a f. r. g. c. s.

3.34 Remark: The converse of proposition (3.33) need not to be true as the following example:

3.35 Example: Let $\tilde{A} = \{(a, 0.9), (b, 0.8), (c, 0.7)\}$ as a universe fuzzy set.

Let $\tilde{B} = \{(a, 0.8), (b, 0.0), (c, 0.0)\}$, $\tilde{C} = \{(a, 0.0), (b, 0.7), (c, 0.0)\}$

$\widetilde{D} = \{(a, 0.8), (b, 0.7), (c, 0.0)\}$ be a fuzzy set of $\tilde{A}$. Then

$\tilde{T} = \{\tilde{\phi}, \tilde{A}, \tilde{B}, \tilde{C}, \widetilde{D}\}$ be a $\mathrm{f.t.s.}$ on $\tilde{A}$.

Take $\widetilde{D} = \{(a, 0.8), (b, 0.7), (c, 0.0)\}$, we are going to explain that $\widetilde{D}$ is a $\mathrm{f.r.}g.\mathrm{c.s.}$ but is not $\mathrm{f.}g.\mathrm{c.s.}$

We notice that $\tilde{A}$ is the only $\mathrm{f.r.o.s.}$ which contains $\widetilde{D}$ and $\tilde{A}$ contains $\widetilde{D}$ too, therefore $\widetilde{D}$ is a f.r.g.c.s. Now since $M\widetilde{D}(x) \leq M\widetilde{D}(x)$ but $\overline{\overline{\widetilde{D}}}(x) = M\tilde{A}(x) \nleq M\widetilde{D}(x)$, therefore $\widetilde{D}$ is not $\mathrm{f.}g.\mathrm{c.s.}$

<u>3.36 Corollaries</u>:

1. Every fuzzy closed set is a f.r.g.c.s.
2. Every f.r.c.s. is a f.r.g.c.s.

<u>3.37 Definition</u>: Let $(\tilde{A}, \tilde{T})$ be a $\mathrm{f.t.s.}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A}, \tilde{T})$, then $\tilde{B}$ is said to be a fuzzy regular generalized open set briefly $(\mathrm{f.r.}g.\mathrm{o.s.})$ if $\tilde{B}^{c}$ is a $\mathrm{f.r.}g.\mathrm{c.s.}$

<u>3.38 Proposition</u>: Every $\mathrm{f.}g.\mathrm{o.s.}$ is a $\mathrm{f.r.}g.\mathrm{o.s.}$

<u>3.39 Remark</u>: The converse of proposition (3.38) need not to be true as the following example:

<u>3.40 Example</u>: In the example (3.35) take $\tilde{B} = \{(a, 0.1), (b, 0.1), (c, 0.7)\}$, it is clear that $\tilde{B}$ is a $\mathrm{f.r.}g.\mathrm{o.s.}$ but not $\mathrm{f.}g.\mathrm{o.s.}$

<u>3.41 Corollary</u>: Every fuzzy open set is a $\mathrm{f.r.}g.\mathrm{o.s.}$

<u>3.42 Remark</u>: The converse of the corollary (3.41) needs not to be true as the following example:

<u>3.43 Example</u>: Let $\tilde{A} = \{(a, 0.9), (b, 0.8), (c, 0.7)\}$ be a universe fuzzy set. Let $\tilde{B} = \{(a, 0.6), (b, 0.2), (c, 0.)\}$, then $\tilde{T} = \{\tilde{\phi}, \tilde{A}, \tilde{B}\}$ be a $\mathrm{f.t.s.}$ on $\tilde{A}$.

Take $\tilde{C} = \{(a, 0.5), (b, 0.1), (c, 0.2)\}$ be a fuzzy set in $(\tilde{A}, \tilde{T})$, it is clear that $\tilde{C}$ in not a fuzzy open set, we are going to show that $\tilde{C}$ is a $\mathrm{f.r.}g.\mathrm{o.s.}$, in other meaning we are going to show that $\tilde{C}^c$ is a $\mathrm{f.r.}g.\mathrm{c.s.}$

Now $\tilde{C}^c = \{(a, 0.4), (b, 0.7), (c, 0.5)\}$ and $M\overline{\tilde{C}^c} = M\tilde{A}(x)$, also all $\mathrm{f.r.o.}$ which contains $\tilde{C}^c$ is $\tilde{A}$, which contains $\overline{\tilde{C}^c}$ too, therefore $\tilde{C}^c$ is a $\mathrm{f.r.}g.\mathrm{c.s.}$ and that means $\tilde{C}$ is a $\mathrm{f.r.}g.\mathrm{o.s.}$ in $(\tilde{A}, \tilde{T})$.

Through above relations among fuzzy closed sets we introduce the following diagram:

f.r.c.s. ⇨ **Fuzzy closed set** ⇨ **f.g^*.c.s.** ⇨ **f.g.c.s.** ⇨ **f.r.g.c.s**

3.44 Definition: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s}$ then $(\tilde{A},\tilde{T})$ is said to be a fuzzy

$\mathrm{T}_{\frac{1}{2}}$ – space briefly $\left(\mathrm{f.T}_{\frac{1}{2}} - \text{ space}\right)$ if every $\mathrm{f}.g.\mathrm{c}.$ in $(\tilde{A},\tilde{T})$ is a fuzzy closed set.

3.45 Definition: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s.}$ then $(\tilde{A},\tilde{T})$ is said to be a fuzzy $\mathrm{R}-\mathrm{T}_{\frac{1}{2}}$ space briefly

$\left(\mathrm{f.R}-\mathrm{T}_{\frac{1}{2}}.\mathrm{s.}\right)$ if every $\mathrm{f.r.}g.\mathrm{c.s.}$ in $(\tilde{A},\tilde{T})$ is a $\mathrm{f.r.c.s.}$

3.36 Theorem: Every $\left(\mathrm{f.R}-\mathrm{T}_{\frac{1}{2}}.\mathrm{s.}\right)$ is a $\mathrm{f.T}_{\frac{1}{2}}.\mathrm{s.}$

3.47 Definition: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s}$ then $(\tilde{A},\tilde{T})$ is said to be a fuzzy $R-$ space briefly $(\mathrm{f.R.s.})$ if every fuzzy closed set in $(\tilde{A},\tilde{T})$ is a $\mathrm{f.r.c.s.}$

3.48 Example: Let $(\tilde{A},\tilde{T})$ be a fuzzy discrete topology, then $(\tilde{A},\tilde{T})$ is a $\mathrm{f.R.s.}$

3.49 Theorem: Every $\mathrm{f.R.T}_{\frac{1}{2}}.\mathrm{s.}$ is a $\mathrm{f.R.s.}$

3.50 Remark: The converse of theorem (3.49) need not be true as the following example:

3.51 Example: Let $\tilde{A}=\{(a,0.9),(b,0.9),(c,0.9)\}$ be a universe fuzzy set.

Let $\tilde{B}=\{(a,0.9),(b,0.0),(c,0.9)\}$, $\tilde{C}=\{(a,0.0),(b,0.9),(c,0.0)\}$ be a fuzzy of $\tilde{A}$, then $\tilde{T}=\{\tilde{\phi},\tilde{A},\tilde{B},\tilde{C}\}$ be a $\mathrm{f.t.s.}$ on $\tilde{A}$.

We are going to show that $(\tilde{A},\tilde{T})$ is a $\mathrm{f.R.s.}$ but is not a $\mathrm{f.R}-\mathrm{T}_{\frac{1}{2}}.\mathrm{s.}$

The fuzzy closed set in $(\tilde{A},\tilde{T})$ are $\tilde{\phi},\tilde{A},\tilde{B},\tilde{C}$ and which are $\mathrm{f.r.c.s.}$ in $(\tilde{A},\tilde{T})$

Now take $\tilde{D}=\{(a,0.0),(b,0.0),(c,0.9)\}$ clearly $\tilde{D}$ is a $\mathrm{f.r.}g.\mathrm{c.s.}$ but is not a $\mathrm{f.r.c.s.}$ in $(\tilde{A},\tilde{T})$, therefore $(\tilde{A},\tilde{T})$ is a $\mathrm{f.R.s}$ but is not a $\mathrm{f.R}-\mathrm{T}_{\frac{1}{2}}.\mathrm{s.}$

4 – Fuzzy semi – generalized closed set

4.1 Definition [M. M. Stadler; M. A. de Parda Vincente,]: Let $(\tilde{A},\tilde{T})$ be a $\mathrm{f.t.s.}$ and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is said to be a fuzzy semi – open set briefly $(\mathrm{f.s.o.s.})$ if there exist a fuzzy open set $\tilde{O}$ in $(\tilde{A},\tilde{T})$ such that

$M\tilde{O}(x)\leq M\tilde{B}(x)\leq M\bar{\tilde{O}}(x)$

4.2 Example: Let $\tilde{A}=\{(a,0.6),(b,0.7)\}$ be a universe fuzzy set, and let $\tilde{B}=\{(a,0.4),(b,0.0)\}$ be a fuzzy set of $\tilde{A}$. Then $\tilde{T}=\{\tilde{\phi},\tilde{A},\tilde{B}\}$ be a $\mathrm{f.t.s.}$ on $\tilde{A}$.

Take $\tilde{C}=\{(a,0.5),(b,0.0)\}$, so $\tilde{C}$ is a $\mathrm{f.s.o.s.}$

4.3 Proposition: Let $(\tilde{A},\tilde{T})$ be a f. t. s. and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is a f. s. o. s. if and only if $M\tilde{B}(x) \le M\bar{\bar{\tilde{B}}}^{\circ}(x)$.

4.4 Definition: Let $(\tilde{A},\tilde{T})$ be a f. t. s., let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is said to be a fuzzy semi – closed set briefly a f. s. c. s. if there exist a fuzzy closed set $\tilde{F}$ in $(\tilde{A},\tilde{T})$ such that $M\tilde{F}^{\circ}(x) \le M\tilde{B}(x) \le M\tilde{F}(x)$

4.5 Proposition: Let $(\tilde{A},\tilde{T})$ be a f. t. s. let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then:

(1) $\tilde{B}$ is f. s. c. s. if and only if $M\bar{\bar{\tilde{B}}}^{\circ}(x) \le M\tilde{B}(x)$.

(2) $\tilde{B}$ is a f. s. c. s. if anf only if $\tilde{B}^{\circ}$ is a f. s. o. s..

4.6 Definition: Let $(\tilde{A},\tilde{T})$ be a f.t.s. and let $\tilde{B}$ be a fuzzy set in $(\tilde{A},\tilde{T})$, then a fuzzy semi – interior briefly $(\text{f.}\bar{\bar{\tilde{B}}}^{\,s})$, $(\text{f.}\tilde{B}^{\circ s})$ respectively, and defined as:

(1) $\text{f.}\bar{\bar{\tilde{B}}}^{\,s} = \inf\{\tilde{G}: \tilde{G} \text{ is a f. s. c. s. in } (\tilde{A},\tilde{T}), M\tilde{B}(x) \le M\tilde{C}(x)\}$

(2) $\text{f.}\tilde{B}^{\circ s} = \sup\{\tilde{H}: \tilde{H} \text{ is a f. s. o. s. in } (\tilde{A},\tilde{T}), M\tilde{H}(x) \le M\tilde{B}(x)\}$

4.7 Definition: Let $(\tilde{A},\tilde{T})$ be a f. t. s. let $\tilde{B}$ a fuzzy set in $(\tilde{A},\tilde{T})$, then $\tilde{B}$ is said to be a fuzzy semi – generalized closed set briefly (f. sg. c. s.) if $Mf.\bar{\bar{\tilde{B}}}^{\,s} \le M\tilde{G}(x)$ wherever $M\tilde{B}(x) \le M\tilde{G}(x)$, where $\tilde{G}$ is a f. s. o. s. in $(\tilde{A},\tilde{T})$.

Now we are going to show that a f. g. c. s. and a f. sg. c. s. are independent concept, namely there is a f. g. c. s. which is not a f. s. c. s. and there is a f. s. c. s. which is not a f. g. c. s. as the following example:

4.8 Examples:

(1) Let $\tilde{A} = \{(a, 0.6), (b, 0.7)\}$ be a universe fuzzy set.

Let $\tilde{B} = \{(a, 0.2), (b, 0.0)\}$, $\tilde{C} = \{(a, 0.4), (b, 0.3)\}$ be a fuzzy sets g $\tilde{A}$.

Then $\tilde{T} = \{\tilde{\phi}, \tilde{A}, \tilde{B}, \tilde{C}\}$ be a f. t. s. on $\tilde{A}$.

Now take $\tilde{D} = \{(a, 0.3), (b, 0.0)\}$, we are going to explain that $\tilde{D}$ is a f. s. c. s. which is not f. g. c. s.

Now $\bar{\bar{\tilde{D}}} = \{(a, 0.4), (b, 0.7)\}$ and $\bar{\bar{\tilde{D}}}^{\circ} = \tilde{\phi}$ so $M\bar{\bar{\tilde{D}}}^{\circ}(x) = 0 \le M\tilde{D}(x)$, therefore $\tilde{D}$ is a f. s. c. s., but is not a f. g. c. s., to explain that:

$M\tilde{D}(x) \le M\tilde{C}(x)$ but $M\bar{\bar{\tilde{D}}}(x) \nleq M\tilde{C}(x)$

(2) Let $\tilde{A} = \{(a, 0.9), (b, 0.9)\}$ be a universe fuzzy set. Let $\tilde{B} = \{(a, 0.6), (b, 0.0)\}$ be a fuzzy set of $\tilde{A}$, then $\tilde{T} = \{\tilde{\phi}, \tilde{A}, \tilde{B}\}$ be a f. t. s. on $\tilde{A}$.

Take $\tilde{C} = \{(a, 0.6), (b, 0.0)\}$ we are going to explain that $\tilde{C}$ is a f. g. c. s. but is not f. s. c. s.

Now $M\tilde{C}(x) \le M\tilde{A}(x)$ and also $M\bar{\bar{\tilde{C}}}(x) \le M\tilde{A}(x)$, therefore $\tilde{C}$ is a f. g. c. s.

Now since $M\bar{\bar{\tilde{C}}}(x) = M\tilde{A}(x)$ and $M\bar{\bar{\tilde{C}}}^{\circ} = M\tilde{A}(x)$

But $M\bar{\bar{\tilde{C}}}^{\circ} \nleq M\tilde{C}(x)$ so $\tilde{C}$ is not a $\mathrm{f.s.c.s.}$

<u>4.9 Remark</u>: Examples (4.8) lead us to conclude that a $\mathrm{f.}g\mathrm{.c.s.}$ and a $\mathrm{f.s}g\mathrm{.c.s.}$ are independent.

References

Mashhour A.s and Ghanim M.H ((Fuzzy closure spaces)) J. math. And App 1.106. pp: 145-170 (1985).

Zedeh L.A. ((Fuzzy sets)), inform control 8, 338-353 (1965).

T. Noir ((Almost – contained and some separation axioms)), Glasnik math. Tm, 9 (24) No. 1-1974, 131- 135.

Levine. N. ((Generalized closed sets in topology)) Rend. Circ. Math. Paleremo (2) 19 (1970), 89- 96.

C.K. Wong ((covering properties of fuzzy topological spaces)), Journal lib of mathematical analysis and application, No. 43, Pages 697- 704.

F. Conrad ((Fuzzy topological concepts)), Journal of mathematical analysis and applications, No, 74, Pages 4323- 440 (1980).

F. G. Sh, ((A new notion of fuzzy compactness in L – Topological spaces)) Information science, No. 173, pages 354 (2005).

M. H. Ghanim; E. E. Kevre, A.S. Mashhour ((Separation Axioms, subspace and sums in fuzzy topology)) Journal of mathematical analysis and applications, No. 102, pages 189- 202 (1984).

M. M. Stadler; M. A. de Parda Vincente, ((Strong separation and strong count ability in fuzzy topological space))m fuzzy sets and systems, No. pages 95- 116 (1991).

R. Lowen, ((A comparison of different compactness notions in fuzzy topological spaces)), Journal of mathematical analysis and applications, No. 64, pages 446- 454 (1978).

C. L. CHANG, ((Fuzzy topological spaces)) 54, 182- 190 (1968).