New Root-Finding Methods for Nonlinear Equations

Eglantina KALLUCI

[1]*University of Tirana, Faculty of Natural Sciences, Tirana-ALBANIA*
eglantina.xhaja@fshn.edu.al

ABSTRACT: In this paper we present three new methods of order four using an accelerating generator that generates root-finding methods of arbitrary order of convergence, based on existing third-order multiple root-finding methods free from the third derivative. The first method requires two-function and three-derivative evaluation per step, and two other methods require one-function and two-derivative evaluation per step. Numerical examples suggest that these methods are competitive to other fourth-order methods for multiple roots and have a higher informational efficiency than the known methods of the same order.

Key-words: root-finding method, multiple root, iterative methods, order of convergence, multiplicity.

1. INTRODUCTION

Solving nonlinear equation is one of the most important problems in numerical analysis. In this paper we consider iterative methods to find a multiple root α. We distinguish with two kinds of methods, those which deal with a known order of multiplicity and others with no information on multiplicity.

We introduce root-finding methods, which are produced by suitable accelerating generators of iterative functions. An iterative method of order $(r+1)$ is generated from the previous method of order (r) using a special transformation. In the following we will present an accelerating formula for generating a sequence of iterative methods for determining multiple roots of equations. Using this generating formula, in Section 2 we present several higher-order iterative methods derived from M. S. Petrović, L. D. Petrović and J. Džunic in their paper (2010) and then using three suitable third order methods given by B. Neta in (Neta, 2008) we will propose three new methods of the fourth order. We will use the well-known theorem from the theory of iterative processes.

Theorem 1 (Traub, 1964, Theorem 2.2)

Let ϕ be an iterative function such that ϕ and its derivatives $\phi', \cdots, \phi^{(r)}$ are continuous in the neighborhood of a root α of a given function f if and only if

$$\phi(\alpha)=\alpha,\phi'(\alpha)=\cdots=\phi^{(r-1)}(\alpha)=0,\phi^{(r)}(\alpha)\neq 0 \quad (1)$$

The following theorem is concerned with the acceleration of iterative methods, forming the base for generating higher-order methods for multiple roots.

Theorem 2 (Petrović et al,2010, Theorem 2)

Let $x_{k+1}=\phi_r(x_k)$ $(k=0,1,\cdots)$ be an iterative method of order (r) for finding a simple or multiple root of a given function f (sufficiently many times differentiable). Then the iterative method derived by

$$x_{k+1}=\phi_{r+1}(x_k)=x_k-\frac{x_k-\phi_r(x_k)}{1-\frac{1}{r}\phi_r'(x_k)}(r\geq 2;k=0,1,\cdots) \quad (2)$$

has order of convergence $(r+1)$. The proof of this theorem is in (Petrović et al, 2010). Further on we will refer to formula (2) as AG(2).

Remark 1. The ability of AG(2) to produce root-finding methods of an arbitrary order of convergence is the main advantage of the generating formula (2). Furthermore, AG(2) can generate iterative formulas both for simple and multiple roots without alternation to its structure; it is sufficient to start with a suitable chosen initial iterative function $\phi_r(x)$ $(r\geq 2)$.

For example, starting from the Newton method

$$\phi_2(x)=x-\frac{f(x)}{f'(x)}=x-u(x)$$

and applying AG(2), we obtain Halley's third-order method

$$\phi_3(x)=x-\frac{f(x)}{f'(x)-\frac{f(x)f''(x)}{2f'(x)}}$$

these methods are applied to simple roots.

We can apply AG(2) to generate iterative methods of an arbitrary order of convergence for finding not only simple roots, but also multiple roots of functions without any modification.

We will consider two classes of methods for finding multiple roots: A- methods where multiplicity is known, B- methods where multiplicity is unknown.

A. Let m be the order of multiplicity of the desired root α of a given function f and let us
introduce the notation

$$u = u(x) = \frac{f(x)}{f'(x)},\ C_r = C_r(x) = \frac{f^{(r)}(x)}{r!\, f'(x)},\ (r = 1,2,\cdots).$$

Starting from the modified Newton method (Schröder,1870) of the second order

$$\phi_2(x) = x - m\frac{f(x)}{f'(x)} = x - mu(x)$$

and using AG(2) is obtained the third-order Halley-like method for multiple roots

$$\phi_3(x) = x - \frac{x - \phi_2(x)}{1 - \frac{1}{2}\phi_2{}'(x)} = x - \frac{mu(x)}{\frac{1}{2}(1+m) - mC_2(x)u(x)}$$

$$\phi_4(x) = x - \frac{mu\left[\frac{1}{2}(1+m) - mC_2u\right]}{\frac{1}{2}(1+m) - mC_2(x)u(x)}$$

$$\phi_5(x) = x - \frac{mu\left[\frac{(1+m)(1+2m)}{3!} - m(1+m)C_2u + m^2C_3u^2\right]}{\frac{1}{4!}(1+m)(1+2m)(1+3m) - \frac{1}{2!}(1+m)(1+2m)C_2u + (1+m)\left[C_3 + \frac{1}{2}C_2{}^2\right]m^2u^2 - m^3C_4u^3}$$

etc.

Remark 2. The above formulas ϕ_4 and ϕ_5 may be regarded as rediscovered formulas since they are originally in (Farmer , Loizou, 1977) using a different approach.

M. S. Petrović, L. D. Petrović and J. Džunic (Petrović et al, 2010) have first derived three new methods:
Using the Chebyshev – like method (Traub, 1977, Petrović et al, 2010)

$$Ch_3(x) = x - mu(x)\left[\frac{3-m}{2} + C_2(x)u(x)\right] \tag{3}$$

of the third order in (2) is obtained the fourth–order method

$$Ch_4(x) = x - \frac{3mu[3 - m + 2mC_2u]}{4 + 3m - m^2 + 6m(m-1)C_2u + 6m^2[C_3 - 2C_2{}^2]u^2}. \quad (4)$$

Using Osada's third–order method (Osada, 1994) in (2) is obtained,

$$O_4(x) = x - \frac{3C_2[(1-m)^2 - 2m(m+1)uC_2]}{4m(m+1)uC_2{}^3 - 6(m+1)C_2{}^2 - 3(m-1)^2C_3}. \quad (5)$$

The third method is obtained by using Ostrowski's third–order method in (2),

$$W_4(x) = x - \frac{3\sqrt{m}u[1 - 2C_2u]}{2[1 - 2uC_2]^{3/2} + \sqrt{m[1 - 3uC_2]} + 3\sqrt{m}u^2C_3}. \quad (6)$$

B. Let α be a multiple root of a function $f(x)$, then α is a simple root of the function $f(x)/f'(x)$. Applying the Newton method to the function $u(x) = f(x)/f'(x)$, we obtain the iterative method

$$\mu_2(x) = x - \frac{u(x)}{1 - C_2(x)u(x)}, \quad (7)$$

which is of the second order. Schröder was the first to derive this method in his paper. Applying $\mu_2(x)$ in (2) is obtained the accelerating method

$$\mu_3(x) = x - \frac{u(x)[1 - 2C_2(x)u(x)]}{1 - 3C_2(x)u(x) + 3C_3(x)u^2(x)}, \quad (8)$$

of order three.

2. NEW METHODS FREE FROM THE THIRD DERIVATIVE

Chebyshev's method for simple roots is given by:

$$\varphi_3(x) = x - u(x)[1 + \frac{1}{2}u(x)C_2(x)]. \quad (9)$$

The two–parameter family is:

$$\varphi_3'(x) = x - \alpha u(x)[1 + \beta u(x) C_2(x)]\,. \tag{10}$$

Neta in (Neta, 2008) has given an estimation how to choose the parameters α and β so that the method is of the third order for the case of multiple roots.
So, for $(m \neq 3)$ and

$$\alpha = -\frac{m(m-3)}{m},\ \beta = -\frac{m}{m-3},$$

we have the third–order Chebyshev's method (3).

The case when $(m = 3)$ needs to be considered separately, however we will restrict our consideration for the cases when $(m \neq 3)$. Also in (Neta, 2008) Neta has developed three new methods not requiring second derivative. The first one is a third–order method based on one–parameter family of modified Chebyshev's method [11]

$$\pi_3'(x) = x - u(x)\left[\beta + \gamma \frac{f(y(x))}{f(x)}\right], \tag{11}$$

where $y(x) = x - \alpha u(x)$, $\alpha = \frac{1}{2}\frac{m(m+3)}{m+1}$, $\beta = m - \frac{m(m-\alpha)}{\alpha^2}$, $\gamma = \frac{m(m-\alpha)}{\rho\alpha^2}$, $\rho = \left(\frac{m-\alpha}{m}\right)^m$.

The second new method is based on the approximation given by Neta (Neta, 2010) for the second derivative in Chebyshev–like method of the third order

$$\pi_3''(x) = x + \frac{m(m-3)}{2} u(x)\left[1 - \frac{m}{m-3} u(x) w(x)\right], \tag{12}$$

where,

$$w(x) = \frac{6(f_{n-1} - f_n) + 2hf_{n-1}' + 4hf_n'}{h^2 f_n'}.$$

The third method is based on the same approximation of the second derivative in Osada's third–order method

$$\pi_3'''(x) = x - \frac{m(m+1)}{2} u(x) + \frac{(m-1)^2}{2w(x)}, \tag{13}$$

where $w(x)$ is given in (12).

Applying the Modified Chebyshev method (11), free from the second derivative to formula (2) we have,

$$\pi_4' = x - \frac{u\left[\beta + \gamma \frac{f(y)}{f(x)}\right]}{\frac{2}{3} + \frac{1}{3}\left[\frac{2C_1C_2 - 1}{2C_1C_2}\left(\beta + \gamma \frac{f(y)}{f(x)}\right) - \frac{f}{f'}\left(f'(y) + f(y)\right)\right]}, \tag{14}$$

which is a first new method of fourth order and requires the evaluation of the second derivative.

The second new method of the fourth order is derived applying the third order one (12) in formula (2)

$$\pi_4'' = x - \frac{x - \pi_3''(x)}{1 - \frac{1}{3}(\pi_3'')'(x)}, \tag{15}$$

where,

$$(\pi_3'')'(x) = -\frac{m(m-3)}{2}\left(1 - 2uC_2\right) + \frac{m^2}{2}\left[(2u - 4u^2C_2)w(x) + u^2\left(v(x) - q(x)\right)\right],$$

$$v(x) = \frac{6\left(f_{n-1}' - f_n'\right) + 2hf_{n-1}''}{h^2 f_n'}, q(x) = \frac{f_n''\left(6\left(f_{n-1} - f_n\right) + 2hf_{n-1}'\right)}{h^2 f_n'^2}.$$

The third new method is derived applying method (13) in formula (2),

$$\pi_4''' = x - \frac{x - \pi_3'''(x)}{1 - \frac{1}{3}(\pi_3''')'(x)}, \tag{16}$$

where,

$$(\pi_3''')' = \frac{m(m+1)}{2}(1-2uC_2) - \frac{(m-1)^2}{2}\frac{v(x)-q(x)}{2w^2(x)}.$$

All three new methods, are of the fourth order and require the evaluation of the second derivative, in contrast with the methods (4), (5) and (6) proposed in (Petrović et al, 2010) that require the evaluation even of the third derivative. In the next Section we will give a table that explains betters the advantages of the methods we have proposed.

3. NUMERICAL TESTS

In this section we give a table comparing the efficiency of six methods for multiple roots including our new ones developed here. The informational efficiency, *E* is defined (Traub, 1964) as

$$E = p/d, \tag{17}$$

and efficiency index, *I*, is defined as

$$I = p^{1/d}, \tag{18}$$

where$p$ is the order of the method and d is the number of function/derivative evaluations per step. Clearly it is assumed that the cost of evaluating a function or any of the derivatives required is identical. In Table 1 we list methods (4), (5), (6), (14), (15) and (16), all of them of the fourth order for finding roots with multiplicity m.

Table 1

Algorithm	p	d	E	I	f'	f''	f'''
(4)	4	4	1	1.414	1	1	1
(5)	4	4	1	1.414	1	1	1
(6)	4	4	1	1.414	1	1	1
(14)	4	5	0.8	1.3195	2	1	-
(15)	4	3	1.333	1.5874	1	1	-
(16)	4	3	1.333	1.5874	1	1	-

The numerical tests of these methods are part of our further work, although we have tested method (14) and it shows global convergence.

4. CONCLUSIONS

In this paper we have proposed three new methods of the fourth order to find the multiple roots of nonlinear equations. The advantage of these methods is that they do not require the evaluation of the third derivative and except of method (14) the informational efficiency and efficiency index are greater than other methods of the same order of convergence. The first new method (14) has informational efficiency and efficiency index smaller than the other methods but, although it shows very good convergence. Numerical tests demonstrate very good convergence behavior even starting from crude initial approximations.

REFERENCES

Buff, X. Hendrikssen, C. (2003) On König's root-finding algorithms nonlinearity. 16, 989-1015.

Kou, J. Li, Y. (2007) Modified Chebyshev's method free from second derivative for nonlinear equations. Appl. Math. Comput. 187, 1027-1032.

Farmer, M. R. Loizou, G. (1977) G. An algorithm for the total, or partial factorization of a polynomial. Math. Proc. Cambridge Philis. Soc. 82, 427-437.

Neta, B. (2008) New third-order nonlinear solvers for multiple roots. Appl. Math. Comput.

Neta, B. (2010) On Popovski's method for nonlinear equations. J. Comput. Appl. Math.

Osada, (1994) N. An optimal multiple root-finding method of order three. J. Comput. Appl. Math. 51, 131-133.

Schröder, E. (1870) ÜberunendlichvielealgorithmenzurAuflösung der Gleichungen. Math. Ann. 2, 317-365.

Petrović, M. S. Petrović L. D., Džunic, J. (2010) Accelerating Generators of Iterative Methods for Finding Multiple Roots. Comput. Math. Appl.

Petrović, M. S. Herceg, C. (1999) On rediscovered iteration method for solving equations. J. Comput. Appl. Math. 107, 275-284.

Traub, J. F. (1964) Iterative Methods for the Solution of Equations. Prentice-Hall, Englewood Cliffs, New Jersey.

Traub, J. F. (1977) Iterative methods for the solution of equations. Chelsea Publication Company, New York.

Vrscay, E. R. Gilbert, W. J. (1998) Extraneous fixed points, basin boundaries and chaotic dynamics for Schröder and Königs rational iteration functions. Numer. Math. 52, 1-16.