0n fuzzy $T_i(i = 0, 1, 2, 3)$ spaces in fuzzy topological groups

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Abstract. The aim of this work is to introduce and study the concepts of fuzzy separation axioms ($fuzzy\ T_0 - spase, fuzzy\ T_1 - spase, fuzzy\ T_2 - spase, fuzzy\ T_3 - spase,$) in fuzzy topological groups and study some theorems and study the relations between these spaces.

Introduction

The concept of fuzzy sets was introduced by zadeh [1]. Chang [2] introduced the definition of fuzzy topological spaces and extended in a straight forward manner some concepts of crisp topological spaces to fuzzy topological spaces. Rosenfeld [3] formulated the elements of a theory of fuzzy groups. A notion of a fuzzy topological group was proposed by foster [4].in this paper we introduce and stady some fuzzy separation axioms in fuzzy $T_i - spaces$, where $i = 0,1,2,3$ in fuzzy topological groups.

Definition (1.1) [L. A. ZADEH , P. E. KLODENK, G. J. KLIR. and Yuan]

If X is a collection of objects with generic element x, then a fuzzy set $\tilde{A}$ in X is characterized by a membership function; $M_{\tilde{A}}$: X $\longrightarrow$ I, where I is the closed unit interval [0, 1], then we write a fuzzy set $\tilde{A}$ by the set of points $\tilde{A} = \{(x, M_{\tilde{A}}(x)) \mid x \in X, 0 \le M_{\tilde{A}}(x) \le 1\}$.
The collection of all fuzzy subsets in X will be denoted by I^X,
i.e $I^X = \{\tilde{A} : \tilde{A}$ is fuzzy set of X$\}$.

Definition (1.2) [C.L.CHANG, B. HUTTON, R. LOWEN]

A fuzzy topology is a family $\tilde{T}$ of fuzzy sets in X, satisfying the following conditions:

(a) $\varnothing, X \in \tilde{T}$.

(b) If $\tilde{A}, \tilde{B} \in \tilde{T}$, then $\tilde{A} \cap \tilde{B} \in \tilde{T}$.

(c) If $\tilde{A}_i \in \tilde{T}$, $\forall$ i $\in$ J, where J is any index set, then $\bigcup_{i \in J} \tilde{A}_i \in \tilde{T}$.

$\tilde{T}$ is called fuzzy topology for X, and the pair (X, $\tilde{T}$) is a fuzzy topological space. Every member of $\tilde{T}$ is called fuzzy open set ($\tilde{T}$-fuzzy open set). A fuzzy set $\tilde{C}$ in X is called fuzzy closed set ($\tilde{T}$-fuzzy closed set) if and only if its complement $\tilde{C}^c$ is $\tilde{T}$- fuzzy open set.

Definition (1.3)[K. K. AZAD, A. MUKHERJEE]

A function f from a fuzzy topological space (X, $\tilde{T}$) into a fuzzy topological space(Y, $\tilde{F}$) is fuzzy continuous function (F-continuous) if and only if the inverse image of each $\tilde{F}$-open fuzzy set is $\tilde{T}$-open fuzzy set.

Definition (1.4) [A. ROSENFELD]

Let X is a group and let $\tilde{G}$ be fuzzy set of X. A fuzzy set $\tilde{G}$ is called a fuzzy group of X if

1- $M_{\tilde{G}}(xy) \geq min\{M_{\tilde{G}}(x), M_{\tilde{G}}(y)\}$ for all $x, y \in$ X .
2- $M_{\tilde{G}}(x^{-1}) \geq M_{\tilde{G}}(x)$ for all $x \in$ X .

Definition (1.5):

A fuzzy group $\tilde{G}$ of a group X is called fuzzy symmetric if $(\tilde{G})^{-1} = \tilde{G}$.

Theorem(1.6):

Every fuzzy group $\tilde{G}$ of X is fuzzy symmetric set.

Proof:

To prove $\tilde{G} = (\tilde{G})^{-1}$, to prove that for every $x \in X$, $M_{\tilde{G}(x)} = M_{(\tilde{G})^{-1}(x)}$.
Since $\tilde{G}$ fuzzy group, then for every $\in X$,
$M_{\tilde{G}}(x) = M_{\tilde{G}}(x^{-1})$
$M_{\tilde{G}}(x) = M_{(\tilde{G})^{-1}}(x)$
Hence $\tilde{G} = (\tilde{G})^{-1}$.

Definition (1.7) [D.H.FOSTER]

Let G be a fuzzy group and $(G, \tilde{T})$ be a fuzzy topological space. $(G, \tilde{T})$ is called a fuzzy topological group if the maps
$g: (G, \tilde{T}) \times (G, \tilde{T}) \rightarrow (G, \tilde{T})$, defined by $g(x, y) = xy$ and
$h: (G, \tilde{T}) \rightarrow (G, \tilde{T})$, defined by $h(x) = x^{-1}$ are fuzzy continuous.

Definition (1.8) [J.KIM]

Let $\tilde{A},\tilde{B}$ be fuzzy sets of G .Then the product $\tilde{A}\tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is the sub set of G and the inverse $\tilde{A}^{-1}$ of $\tilde{A}$ is the sub set of G by respectively formules , $M_{\tilde{A}\tilde{B}}(x) = \sup\{min\{M_{\tilde{A}}(y), M_{\tilde{B}}(z): y, z = xand$

$M_{\tilde{A}^{-1}}(x) = M_{\tilde{A}}(x^{-1})$ for all $x \in G$.

Definition (1.9):

A fuzzy set in a fuzzy topological group (G , $\tilde{T}$) is called fuzzy neighborhood of a fuzzy point x in G if there is a fuzzy open set $\widetilde{U}$ in G ,such that $x \in \widetilde{U} \subseteq G$.

Definition (1.10):

A fundamental system of fuzzy neighborhood of $\tilde{e}$ in (G, $\tilde{T}$) is a collection $\{\widetilde{U}\}$ of fuzzy neighborhood of $\tilde{e}$ such that every fuzzy neighborhood of $\tilde{e}$ contains a member of $\{\widetilde{U}\}$. If each member of $\{\widetilde{U}\}$ is fuzzy open, we said of a fundamental system of fuzzy open neighborhood of $\tilde{e}$.

Theorem(1.11):

Let (G,$\tilde{T}$) be a fuzzy topological group, then there exists a fundamental system $\{\widetilde{U}\}$ of fuzzy symmetric neighborhood of $\tilde{e}$.

Proof:

Let $\{\tilde{V}\}$ is a fundamental system of fuzzy open neighborhood of $\tilde{e}$.
Since $\tilde{e} = \tilde{e}^{-1}$ by theorem (1.6) .
Shows that for each $\tilde{V}$ in $\{\tilde{V}\}$, $\tilde{V}^{-1}$ is an fuzzy open neighborhood of $\tilde{e}$.
But $M_{\widetilde{U}}(x) = min\,\{ M_{\tilde{V}}(x), M_{\tilde{V}^{-1}}(x)\}$ is a fuzzy symmetric neighborhood of $\tilde{e}$, because
$M_{\widetilde{U}^{-1}}(x) = min\,\{ M_{\tilde{V}}(x), M_{\tilde{V}^{-1}}(x)\} = M_{\widetilde{U}}(x)$.
Therefore, each $\tilde{V}$ contains a $\widetilde{U}$.
On the other hand, each fuzzy neighborhood of $\tilde{e}$ contains a $\tilde{V}$ and so $\{\widetilde{U}\}$ is a fundamental system of fuzzy symmetric neighborhood of $\tilde{e}$.

Definition (1.12):

A fuzzy topological group (G, $\tilde{T}$) is said to be

1- Fuzzy $\tilde{T}_0$ – space if for any distinct fuzzy points $\tilde{p}\,,\tilde{q}$ in G , there exists a fuzzy neighborhood $\widetilde{U}$ in G such that $\tilde{p} \in \widetilde{U}\,, \tilde{q} \notin \widetilde{U}$ or $\tilde{q} \in \widetilde{U}\,, \tilde{p} \notin \widetilde{U}$.
2- Fuzzy $\tilde{T}_1$ – space if for any distinct fuzzy points $\tilde{p}\,,\tilde{q}$ in G, there exists a fuzzy neighborhoods $\widetilde{U},\tilde{V}$ in G such that $\tilde{p} \in \widetilde{U}\,, \tilde{q} \notin \widetilde{U}$ and $\tilde{q} \in \tilde{V}, \tilde{p} \notin \tilde{V}$.
3- Fuzzy $\tilde{T}_2$ – space (fuzzy Hausdorff $-$space) if for any distinct fuzzy points $\tilde{p}\,,\tilde{q}$ in G, there exists fuzzy neighborhoods $\widetilde{U},\tilde{V}$ in G such that $\tilde{p} \in \widetilde{U}\,, \tilde{q} \notin \widetilde{U}$ and $\tilde{q} \in \tilde{V}\,, \tilde{p} \notin \tilde{V}$ such that $\widetilde{U} \cap \tilde{V} = \emptyset$.
4- Fuzzy $regular$ space if $\tilde{p} \in$ G and a closed fuzzy set $\tilde{F} \subset$G with $\tilde{p} \notin \tilde{F}$ $\exists$ fuzzy neighborhoods $\widetilde{U}$and $\tilde{V}$ s.t $\tilde{p} \in \widetilde{U}$, $\tilde{F} \subset \tilde{V}$ and $\widetilde{U} \cap \tilde{V} = \emptyset$
5- Fuzzy $\tilde{T}_3$ $-$ space if G are Fuzzy $\tilde{T}_1$ – space and Fuzzy regular space.

Theorem(1.13):

Let (G,$\tilde{T}$) be a fuzzy topological group (G,$\tilde{T}$), then

1- Every fuzzy T_3 $-$topological group is a fuzzy Hausdorff $-$space.

2- Every fuzzy Hausdorff $-$topological group is a fuzzy T_1 $-$space.

3- Every fuzzy T_1 $-$topological group is a fuzzy T_0 $-$space.

Proof:

Obvious.

Theorem(1.14):

Every fuzzy T_0 $-$topological group is a fuzzy T_1 $-$space .

Proof:

Let (G,$\tilde{T}$) be a fuzzy topological group .

Let $\tilde{p} \neq \tilde{q}$, $\tilde{p}, \tilde{q} \in G$, there exists an fuzzy open neighborhood $\tilde{U}$ of $\tilde{p}$ such that $M_{\tilde{q}}(x) > M_{\tilde{U}}(x)$.

Since $M_{\tilde{p}^{-1}\tilde{U}}(x) = M_{\tilde{V}}(x)$ is an fuzzy open neighborhood of $\tilde{e}$, $min\,\{M_{\tilde{V}}(x), M_{\tilde{V}^{-1}}(x)\} = M_{\tilde{W}}(x)$ is fuzzy open symmetric neighborhood of $\tilde{e}$ and therefore $\tilde{q}\tilde{W}$ is a fuzzy neighborhood of $\tilde{q}$.

Now $M_{\tilde{p}}(x) > M_{\tilde{q}\tilde{W}}(x)$ because otherwise $M_{\tilde{p}^{-1}}(x) \leq M_{\tilde{W}\tilde{q}^{-1}}(x)$ and , hence , $M_{\tilde{p}^{-1}}(x) \leq M_{\tilde{W}\tilde{q}^{-1}}(x) \leq M_{\tilde{V}\tilde{q}^{-1}}(x) \leq M_{\tilde{p}^{-1}\tilde{U}\tilde{q}^{-1}}(x)$

But this implies that $M_{\tilde{e}}(x) = M_{\tilde{p}\,\tilde{p}^{-1}}(x) \leq M_{\tilde{p}\,\tilde{p}^{-1}\tilde{U}\tilde{q}^{-1}}(x) = M_{\tilde{U}\tilde{q}^{-1}}(x)$, or $M_{\tilde{q}}(x) \leq M_{\tilde{U}}(x)$,which is contradiction.

Theorem(1.15):

Every fuzzy T_1 $-$topological group is a fuzzy Hausdorff $-$space.

Proof:

Let (G,$\tilde{T}$) be a fuzzy topological group.

Let $\tilde{p} \neq \tilde{q}$, $\tilde{p}, \tilde{q} \in G$,

∵ G is a fuzzy T_1 $-$space then $\{\tilde{p}\}$ is a fuzzy closed set and therefore

$\tilde{U} \in G$ and $\tilde{U} \notin \{\tilde{p}\}$ is an fuzzy open neighborhood of $\tilde{q}$ and hence $\tilde{q}^{-1}\tilde{U}$ is an fuzzy open neighborhood of $\tilde{e}$.

Let $\tilde{V}$ is an fuzzy open neighborhood of $\tilde{e}$,

such that $M_{\tilde{V}\tilde{V}^{-1}}(x) \leq M_{\tilde{q}^{-1}\tilde{U}}(x)$.

Then $\tilde{q}\tilde{V}$ is an fuzzy open neighborhood of $\tilde{q}$.

Let $\tilde{W} \in G$ and $M_{\tilde{W}}(x) > M_{\overline{\tilde{q}\tilde{V}}}(x)$ which is an fuzzy open set.

And $M_{\tilde{p}}(x) \leq M_{\tilde{W}}(x)$.

For otherwise $M_{\tilde{p}}(x) > M_{\overline{\tilde{q}\tilde{V}}}(x)$ and, hence,

$min\,\{M_{\tilde{p}\tilde{V}}(x), M_{\tilde{q}\tilde{V}}(x)\} \neq \emptyset$.

But this shows that $M_{\tilde{p}}(x) \leq M_{\tilde{q}\tilde{V}\tilde{V}^{-1}}(x) \leq M_{\tilde{q}(\tilde{q}^{-1}\tilde{U})}(x) = M_{\tilde{U}}(x)$, which is contradiction because $\tilde{p} \notin \tilde{U}$.

Clearly $\{M_{\tilde{W}}(x), M_{\tilde{q}\tilde{V}}(x)\} = \emptyset$, $M_{\tilde{q}}(x) \leq M_{\tilde{q}\tilde{V}}(x)$ and $M_{\tilde{p}}(x) \leq M_{\tilde{W}}(x)$, $\tilde{q}\tilde{V}$ and $\tilde{W}$ are fuzzy open sets .

Theorem(1.16):

Every fuzzy topological group is a fuzzy regular $-$space.

Proof:

Let (G,$\tilde{T}$) be a fuzzy topological group.

By homogeneity it is enough to show that if $\tilde{F}$ is a fuzzy closed in G and $M_{\tilde{e}}(x) > M_{\tilde{F}}\ (x)$ then there exists fuzzy open sets $\tilde{U}, \tilde{V}$ with $M_{\tilde{F}}(x) \leq M_{\tilde{U}}(x)$, $M_{\tilde{e}}(x) \leq M_{\tilde{V}}(x)$ and $min\ \{M_{\tilde{U}}(x), M_{\tilde{V}}(x)\} = \emptyset$.

Now the complement of $\tilde{F}$ is neighborhood of $\tilde{e}$;

We can therefore find an open neighborhood $\tilde{V}$ of $\tilde{e}$ such that $min\ \{\ M_{\tilde{V}\tilde{V}^{-1}}(x), M_{\tilde{F}}(x)\} = \emptyset$.

But this implies $min\ \{\ M_{\tilde{V}}(x), M_{\tilde{V}\tilde{F}}(x)\} = \emptyset$, so we may take $M_{\tilde{U}}(x) = M_{\tilde{V}\tilde{F}}\ (x)$ which contains $\tilde{F}$and is fuzzy open set.

Corollary(1.17):

1- Every fuzzy T_1 –topological group is a fuzzy T_3 –space.
2- Every fuzzy Hausdorff –topological group is a fuzzy T_3 –space.

Proof:

Obvious.

Theorem(1.18):

Let (G,$\tilde{T}$) be a fuzzy Hausdorff topological group, then $\cap\{\tilde{U}\} = \tilde{e}$, where $\{\tilde{U}\}$ is a fundamental system of fuzzy neighborhood of $\tilde{e}$.

Proof:

Let $\tilde{p} \in \tilde{U}$ for each $\tilde{U}$ in $\{\tilde{U}\}$ and assume $\tilde{p} \neq \tilde{e}$.

∵ G is fuzzy Hausdorff –space, then implies that there exists an fuzzy open neighborhood $\tilde{V}$ of $\tilde{e}$ such that$M_{\tilde{p}}(x) > M_{\tilde{V}}\ (x)$.

But then there exists a $\tilde{U}$ in $\{\tilde{U}\}$ such that $M_{\tilde{U}}(x) \leq M_{\tilde{V}}(x)$.

We have the contradiction:

$M_{\tilde{p}}(x) \leq M_{\tilde{U}}(x) \leq M_{\tilde{V}}(x)$ and $M_{\tilde{p}}(x) > M_{\tilde{V}}(x)$.

Hence, $\tilde{p} = \tilde{e}$.

Theorem(1.19):

Let (G,$\tilde{T}$) be a fuzzy topological group, if $\cap\{\tilde{U}\} = \tilde{e}$, then (G,$\tilde{T}$) is fuzzy T_o –space .

Proof:

Let $\tilde{p} \neq \tilde{q}$, $\tilde{p}, \tilde{q} \in G$,

Then $M_{\tilde{p}\tilde{q}^{-1}}(x) \neq M_{\tilde{e}}\ (x)$ and, hence

$\because \cap\tilde{U} = \tilde{e}$ Then there exists a $\tilde{U}$ in $\{\tilde{U}\}$ such that $M_{\tilde{p}\tilde{q}^{-1}}(x) > M_{\tilde{U}}(x)$, thus $\tilde{U}\tilde{q}$ being a fuzzy neighborhood of $\tilde{q}$ and $M_{\tilde{p}}(x) > M_{\tilde{U}\tilde{q}}(x)$.

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