# RELATIVE CONTROLLABILITY OF NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL SYSTEMS IN ABSTRACT SPACE WITH DISTRIBUTED DELAYS IN THE CONTROL.

**By**

**Oraekie, Paul. Anaetodike., Ph.D**

**Department of Mathematics, Anambra State University, Uli,**

**Anambra State of Nigeria, West Africa.**

**E-mail: paul.oraekie@yahoo.com**

**Tel: 07031982483**

**Abstract**

In this paper, Abstract Neutral Functional Integrodifferential System with Distributed Delays in the control was presented for relative controllability analysis. We used variation of parameters to obtain the solution of our system of interest as an integral equation. The integral equation contains the values of the control $u(t)\ for\ t < 0\ as\ wellas\ for\ t > 0$ .The values of the control u(t) for $t\epsilon[-h, 0]$ enter into the definition of the initial complete state. To separate them we applied the Unsymmetric Fubini Theorem and the integration is in the Lebesque-Stielties sense. The set functions (controllability grammian, reachable set, attainable set, target set) upon which our study hinges were extracted and thus established that the system is relatively controllable. However, necessary and sufficient conditions for the relative controllability of the system were stated and established/proved.

**Key Words:**

Unsymmetric Fubini theorem, Target set, Controllability map, Properness, Controllability, Reachable set, and Attainable set.

## 1. INTRODUCTION:

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations infinite dimensional space has been extensively studied. Many authors have extended the controllability concept to infinite dimensional systems in Bamach spaces with bounded operators. K. Naito (1989) has studied the controllability of semi linear systems. M. Yamamoto and J. Y. Park (1990) discussed the same problem for parabolic equation with uniformly bounded nonlinear term. While E.N. Chukwu and Lenhart (1991) studied the controllability of nonlinear systems in abstract spaces. M. D. Quinn and Carmichael(1984-1985) showed that the controllability problem in Banach space can be converted into a fixe pointed problem for a single-valued mapping. Balachandran (1996, 1998) had studied the controllability and local null controllability of Sobolve-type integrodifferential systems in Banach spaces by using Schauder's fixed point theorem.

The purpose of this work is to investigate the relative controllability of the following abstract neutral functional integrodifferential system with distributed delays in the control.

$$\frac{d}{dt}[x(t)-g(t,x_t)]+Ax(t)=\int_0^t f(s,x_s)ds+\int_{-h}^0(d_s\,C(s,l)u(s+l) \qquad (1.1)$$

$$x(t_0)=x_0=\emptyset\in B,\ t\in[0,a]=J.$$

Where $B$ is the phase space, the state variable $x(.)$takes values in Banach space X and the control function $u(.)$ is given in $L_2(J,U)$, $(where\ |u_j|\leq 1, j=1,2,\dots.m)$, the Banach space of admissible control functions with $U\ a$ Banach space. $C$ is a bounded linear operator from $U$ into $X$, the unbounded linear operators $-A$ generates an analytic semi-group, and $f,g\colon JxB\to X$ are appropriate functions.

### 2.0 Preliminaries and Definitions

Throughout this work X will be a Banach space with norm $\|.\|$, $-A\colon D(A)\to X$ will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $T(t)$. Let $0\in P(A)$,then it is possible to define the fractional power $A^\alpha, for\ 0<\alpha\leq 1$, as a closed linear operator on its domain D($A^\alpha$). To study the system (1.1), we assume that the histories $x_t\colon(\infty,0]\to X, x(_t\theta)=x(t+\theta)$ belong to some abstract phase space B, which is defined axiomatically. In this work, we will employ an axiomatic definition of the phase space B introduced by Hale and Kato (1978) and follow the terminology used in the paper of Y. Hino (1991). Thus, B will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a norm $\|.\|_B$ .Let us assume that B satisfies the following axioms:

(1) If $x\colon[-\infty,\alpha\propto+a]\to x,\ a>0$, is continuous on $[\propto,\propto+a]\ and\ x_\propto\epsilon\ B$, then for every time $\tau\in[\propto,\propto+a]$ the following conditions hold:

(a) $x_t\ is\ in\ B$;

(b) $\|x(t)\| \leq K \|x_\tau\|_B$;

(c) $\|x_\tau\|_B \leq H(\tau-\propto) sup\{\| x(s)\| : \propto \leq s \leq \tau\} + M(\tau-\propto)\|x_\propto\|_B$.

$Here\ k > 0\ is\ a\ constant, H, M: [0,\infty) \to (0,\infty).$

H is continuous and M is locally bounded, and K, H, M is independent of $x(t)$.

(d) For the function $x\ (.)in\ (1), x_t$ is a B-valued continuous function on $[\propto, \propto + a]$;

(2) The space B is a complete space. Now we can give basic assumptions on the system (1.1).

(i) $g: [0,a]\ x\ B \to X$ is a continuous function, and there exists a constant $\lambda \in (0,1)$ and $P, P_1 > 0$, such that the function g is $X_\lambda$ −valued and satisfies the lipschitz condition:

$$\| A^\lambda g(t_1, \emptyset_1) - A^\lambda g(t_2, \phi_2)\| \leq P(|t_1 - t_2| + \|\phi_1 - \phi_2\|_B),$$

$$for\ 0 \leq t_1, t_2 \leq a; \phi_1, \phi_2 \epsilon B, and\ the\ inequality$$

$$\|A^\lambda g(t,\phi)\| \leq P_1(\|\phi\|_B + 1) holds\ for\ t \in J = [0,a], \phi \in B.$$

(3) The function $f: [0,a]xB \to X$ satisfies the following conditions:

(i) $For\ each\ t \in J, th\ function\ f(t,.): B \to X\ is\ continuous$

$and\ for\ each\ \phi \in B, the\ function\ f(.,\phi): J \to X\ is\ strongly\ measurable,$

(ii) $For\ each\ positive\ number\ n, there\ is\ a\ positive\ function\ \propto \in L_1([0,a])$

$such\ that$ $\text{Sup}_{\|\phi\| \leq n} \|f(t,\phi)\| \leq \alpha_n(t)$

and

$$\liminf_{n \to \infty} \frac{1}{n} \int_0^a \int_0^t \propto_n (s) ds\ dt = \gamma < \infty$$

(4) The linear operator $W$ from $U$ into $X$ is defined by

$w_u = \int_0^a T(t-s)[\int_{t-h}^0 d_s\ C(s,l)]\ u(s+l) ds$

and there exists a bounded invertible operator $w^{-1}\ defined\ in$

$L_2(J;\ U)/\ker w.$,

where $C$ is a bounded linear operator.

**2.01: Variation of parameters**

The function $x(.): (-\infty, a) \to x$ is a solution of system (1.1) if $x_0 = \phi$, then the restriction of $x(.)$ to the interval $[0,a]$ is continuous and for each $0 \leq t \leq a$, the function $AT(t-s)g\ (s, x_s), s \in [0,t]$ is integrable and the following integral equation is the required solution of system (1.1).

$$x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_s) - \int_0^t AT(t-s) g(s,x_s)$$
$$+ \int_0^t T(t-s)[\int_{-h}^0 (d_s\ C(s,l) u(s+l) + \int_0^s f(\tau, x_\tau) d\tau]\ ds, t \in J \quad (2.1)$$

$$(2.1)\Rightarrow x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) - \int_0^t AT(t-s)g(s,\ x_s)ds$$
$$+\ \int_0^t T(t-s)[\int_{-h}^0 (d_s\,C(s,l))u(s+l)]ds + \int_0^t T(t-s\_[\int_0^s f(\tau,x_\tau)d\tau]\,ds \qquad (2.2)$$

The fourth term in the right-hand side of system (2.2) contains the values of the control $u(t)$ for $t < 0, as\ well\ as\ for\ t > 0,$ the values of the control $u(t) for$ $t \in [0-h,0]$ enter into the definition of the initial complete state $y_{t_0}$ . To separate them, the fourth term of system (2.2) must be transformed by changing the order of integration. Using the **unsymmetric Fubini Theorem**, we have the following equalities:

$$x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)ds$$
$$+\int_{-h}^0 dc_s\,(\int_0^t \mathrm{T(t-s)C(s,l)u(s+l)ds}) + \int_0^t T(t-s)[\int_0^s f(\tau,x\tau)d\tau]\,ds \qquad (2.3)$$
$$= T(t)[\emptyset(0) - g(0,\phi)] + g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)\,ds$$
$$+\int_0^t T(t-s)[\int_0^s f(\tau,x_\tau)d\tau]ds$$
$$+\int_{-h}^{0} dc_s\,(\int_{0+\mathrm{l}}^{\mathrm{t+l}} \mathrm{T(t-s)C(s-l,l)u(s-l+l)ds}$$
$$= T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) -\ \int_0^t AT(t-s)g(s,x_s)ds$$
$$+\int_0^t T(t-s)\int_0^s f(\tau,x_\tau)d\tau ds$$
$$+\int_{-h}^{0} dc_s\,(\int_{0+\mathrm{l}}^{0} \mathrm{T(t-s)C(s-l,l)u_0}\,(s)ds)$$
$$+\int_{-h}^0 dc_s\,(\int_0^{\mathrm{t+l}} \mathrm{T(t-s)\ C(s-l,l)u(s)ds}) \qquad (2.4)$$

where $dc_s$ denotes that the integration is in the **Lebesque-Stielties** sense with respect to the variable $s$ in the function $C(s,l)$.

Let us introduce the following notation:

$$C(s,l) = \begin{Bmatrix} C(s,l), for\ s < t, l \in R \\ 0 \quad ,for \quad s > t,\ \ l \in R \end{Bmatrix} \qquad (2.5)$$

Hence $x(t)$ can be expressed in the following form:

$$x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)ds$$
$$+\int_0^t T(t-s)\int_0^s f(\tau,x_\tau)d\tau\,ds + \int_{-h}^0 dc_s\,(\int_{0+\mathrm{l}}^0 \mathrm{T(t-s)C(s-l,l)u_0(s)}\,ds)$$
$$+\int_{-h}^0 dc_s\,(\int_0^{\mathrm{t}} \mathrm{T(t-s)\hat{C}(s-l,l)u(s)ds}) \qquad (2.6)$$

Using again the **unsymmetric Fubini Theorem,** the equality (2.6) can be rewritten in more convenient form as follows:

$$x(t) = T(t)[\phi(0) - g(0,\emptyset)] + g(t,x_t) - \int_0^t AT(t-s)g(s,x_s)ds$$
$$+\int_0^t T(t-s)\int_0^s f(\tau,x_\tau)d\tau\,ds + \int_{-h}^0 dc_s\int_{-\mathrm{h}}^0 \mathrm{T(t-s)C(s-l,l)u_0}\,(s)\mathrm{ds}$$

$$+\int_0^t \left(\int_{-h}^0 T(t-s)d_s\,\hat{C}(s-l,l)u(s)ds\right) \quad (2.7)$$

Now let us consider the solution $x(t)$ of system (1.1) for $t = t_1 = a$

$$x(t_1) = T(t_1)[\phi(0) - g(0,\phi)] + g\left(t_1, x_{t_1}\right) - \int_0^{t_1} AT(t_1 - s)\, g(s, x_s)\, ds$$

$$+\int_0^{t_1} T(t_1 - S)\int_0^S (\tau, X_\tau)\, d\tau ds + \int_{-h}^0 dc_s\left(\int_{0+l}^0 \mathrm{T}(\mathrm{t}-\mathrm{s})\mathrm{C}(\mathrm{s}-\mathrm{l},\mathrm{l})\mathrm{u}_0(\mathrm{s})\mathrm{ds}\right.$$

$$+\int_0^{t_1}\left(\int_{-h}^0 T(t_1 - s)\, d_s\hat{C}(s-l,l)\right)u(s)ds \quad . \qquad (2.8)$$

Consider system (2.7), for brevity, let,

$$\beta(t) = T(t)\,[\phi(0) - g(0,\phi)] + g(t, x_t) - \int_0^t AT(t-s)g(s,x_s)ds \qquad (2.9)$$

$$\mu(t) = \int_0^t T(t-s)\int_0^s f(\tau, x_\tau)d\tau\, ds$$

$$+\int_{-h}^0 dc_s \int_{0+l}^0 \mathrm{T}(\mathrm{t}-\mathrm{s})\mathrm{C}(\mathrm{s}-\mathrm{l},\mathrm{l})\mathrm{u}_0(\mathrm{s})\mathrm{ds}\,. \qquad (2.10)$$

$$\mathfrak{z}z(t,s) = \int_{-h}^0 T(t-s)d_s\,\hat{C}(s-l,l) \qquad (2.11)$$

Substituting (2.9), (2.10), (2.11) in (2.7), we have a precise variation of constant formula for system (1.1) as

$$x(t, x_0, u) = \beta(t) + \mu(t) + \int_0^t \mathfrak{z}z(t,s)u(s)ds \qquad (2.12)$$

**Definition 2.1 (Complete State)**

The complete state for system (1.1) is given by the set $\boldsymbol{g(t)} = \{\boldsymbol{x}, \boldsymbol{u_t}\}$.

**Definition 2.2_ (Relative Controllability)**

The system (1.1) is said to be relatively controllable on $[0,a]$ if for every initial complete state $g(0) and x_1 \in X$ there exists a control function $u(t)$defined on $[0,a]$ such that the solution of system (1.1) satisfies $x(t_1) = x_1$).

**2.02: Basic Set Functions and Properties.**

**Definition 2.3 (Reachable set)**

The reachable set for the system (1.1) is given as

$$R(t_1, 0) = \{\int_0^{t_1} [\int_{-h}^{o} T(t_1 - s)d_s\hat{\mathrm{C}}\,(\mathrm{s}-\mathrm{l},\mathrm{l})]\ \mathrm{u(s)\ ds}\}$$

**Definition 2.4_ (Attainable set)**

The attainable set for the system (1.1) is given as

$$A(t,0) = \{x(t, x_0, u)x : u \in U\},$$
$$where\ U = \{u \in L_2([0,a]; X) : |uj| \le 1; j = 1,2,\ldots,m\}$$

**Definition 2.5 (Target set)**

The target set for system (1.1) denoted by $G(t_1, 0)$ is given as

$$G(t_1,0) = \{x(t_1, x_0, u) : t_1 \ge \tau > 0\ for\ fixe\ \tau\epsilon J\ and\ u \in U\}$$

**Definition 2.6 (Controllability grammian**)

The controllability grammian of the system (1.1) is given as

$$W(t_1,0) = \int_0^{t_1,} (\int_{-h}^{0} T(t_1 - s) d_s \hat{\mathrm{C}}\,(\mathrm{s} - \mathrm{l}, \mathrm{l}) ( \int_{-h}^{0} T(t_1 - s) d_s \hat{\mathrm{C}}\,(\mathrm{s} - \mathrm{l}, \mathrm{l}))^T$$

Where T denotes matrix transpose.

**2.03: Relationship between the Set Functions**

We shall first establish the relationship between the attainable set and the reachable set to enable us see that once a property has been proved for one set, and then it is applicable to the other.

From equation (2.7),

$$A(t,0) = \eta(t) + R(t,0\,) for\, u \in U, t \in [0,a],$$
$$where\, \eta(t) = \beta(t) + \mu(t).$$

This means that the attainable set is the translation of the reachable set through $\eta \in X$. Using the attainable set, therefore, it is easy to show that the set functions possess the properties of convexity, closeness and compactness.

Also, the set functions are continuous on $[0,\infty]$ to the metric space of compact subject of $X = E^n$. E.N. Chukwu (1988) and I.Gyori (1982) give impetus for adaptation of the proofs of these properties for system (1.1).

**Definition 2.7 (Properness)**

The system (1.1) is proper in $X = E^n\ on\ [0,a]\ if\ span\ R(t,0) = X = (E^n)$

i.e if

$$\mathrm{C}^{\mathrm{T}}[\int_{-\mathrm{h}}^{0} \mathrm{T}(\mathrm{t}-\mathrm{s})\mathrm{d}_{\mathrm{s}} \hat{C}(s-l,l)] = 0 \quad a.e.$$
$$a > 0 \Longrightarrow c = 0; c \in X = E^n$$

***3. The Main Result***

**Theorem 3.1** Consider the system (1.1) given as

$$\frac{d}{dt}(x(t) - g\,(t,x_t)) + Ax(t) = \int_0^t f(s,x_s)\,ds + \int_{-h}^0 d_s\, C(s,l)u(s+l) \qquad (3.1)$$

$x(t_0) = x_0 = \phi \in B; t \in [0,a] = J$

with its standing hypothesis, then the following statements are equivalent:

(i) System (3.1) is relatively controllable on$[0,a]$.

(ii) The controllability grammian $w(t,0)$ of system (3.1) is non-singular.

(iii) System (3.1) is proper on $[0,t_1]$ .

**Proof**

From the controllability standard, we realized that: $w(t,0)$is non-singular, is equivalent to $w(t,0)$ is positive definite, which in turn is equivalent to $C^T$ times the controllability index $= 0$.

The controllability index for this system (3.1) is given as

$$[\int_{-\mathrm{h}}^{0} \mathrm{T}(\mathrm{t}-\mathrm{s})\mathrm{d}_{\mathrm{s}} \hat{C}(s-l,l)] = 0 \qquad a.e\ \ on\,[0,a] \quad \Rightarrow C = 0$$

Thus (ii) and (iii) are equivalent.

To prove that (i) and (iii) are equivalent.

$$Let\ C \in X = E^n (where\ E^n is\ Euclidean\ space)\ \ and\ assume\ that$$

$$C^T[\int_{-h}^{0} T(t-s)d_S\,\hat{C}(s-l,l) = 0 \quad a.e, \qquad t \in [0,a], for\ each\ t, then$$

$\int_0^t C^T\,[\int_{-h}^{0} T(t-s)d_s\,\hat{C}(s-l,l)]\,u(s)\,ds$

$$= C^T\int_0^t\int_{-h}^{0} T(t-s)d\,s\,\hat{C}(s-l,l)]\,u(s)\,ds\,,\ \ for\ u \in L_2$$

It follows that C is orthogonal to the reachable set $R(t,0)$, where

$$R(t,0) = \{\int_0^t\int_{-h}^{0} T(t-s)d\,s\,\hat{C}(s-l,l)]\,u(s)\,ds\,,\ \ for\ u \in L_2\}$$

If we assume the relative controllability of system (1.1) visa vis (3.1), then $R(t,0) = X = E^n$ so $that\ c = 0, showing\ that\ \ (i) \Longrightarrow (iii)$.

Conversely,

Assume the system (3.1) is not controllable, so that the reachable set $R(t,0) \neq X\,, for\ t > 0$. Then there exists $c \neq 0, c \in X = E^n$ such that $C^T R(t,0) = 0$

$$It\ follows\ that\ for\ all\ admissible\ control\ u \in L_2\ \ that$$

$$0 = C^T\int_0^t[\int_{-h}^{0} T(t-s)d_s\,\hat{C}(s-l,l)u(s)ds$$

$$= \int_0^t C^T\,[\int_{-h}^{0} T(t-s)d_s\,\hat{C}(s-l,l)]\,u(s)\,ds$$

$$Hence,$$

$C^T[\int_{-h}^{0} T(t-s)d_S\,\hat{C}(s-l,l) = 0 \quad a.e,$

$t \in [0,a], \Rightarrow c \neq 0$

**Theorem 3.2**

**(Necessary and sufficient conditions for the system to be relatively controllable)**

The system (3.1) with its standing hypothesis is relatively controllable if and only if zero is in the interior of the reachable set of the system for $a > 0, J = [0,a]$.

**Proof**

The need to show that $0 \in interior\ R(t,0)$. The reachable set $R(t,0)$ is a closed and convex subset of X. Therefore, a point $z_1 \in X$ on the boundary implies there is a support plane $\boldsymbol{\pi}$ of $R(t,0)$ through $2z_1$. That is,

$$C^T(2z - z_{1)}2 \leq 0, for\ each\ z \in R(t,0).$$

Where $C \neq 0$ is an outward normal to the support plane $\pi$.

If $u$, is the corresponding control to $2z_{1,}$we have

$C^T\int_0^t[\int_{-h}^{0} T(t-s)d_s\,\hat{C}(s-l,l)u(s)ds$

$$\leq\ C^T\int_0^t[\int_{-h}^{0} T(t-s)d_S\,\hat{C}(s-l,l)u(s)ds \qquad (3.2)$$

For each $u \in U$. Since $U$ is a unit sphere the inequality (3.2) becomes.

$$|C^T\int_0^t[\int_{-h}^{0} T(t-s)d_s\hat{C}\,(s-l,l)u(s)ds| \leq \int_0^t |C^T[\int_{-h}^{0} T(t-s)d_S\hat{C}\,(s-l,l)].1|ds$$

$$= \int_0^t C^T [\int_{-h}^0 T(t-s) d_S \hat{C}(s-l,l)] \, \boldsymbol{sgn} \, C^T [\int_{-h}^0 T(t-s) d_S \hat{C}(s-l,l) \quad (3.3)$$

Comparing system (3.2) and (3.3), we have

$$u_1(t) = \boldsymbol{sgn} C^T [\int_{-h}^0 T(t-s) d_s \hat{C}(s-l,l)] \quad (3.4)$$

More so, as $2z_1$ is on the boundary, since we always have $0 \in R(t,0)$. If 0 were not in the interior of $R(t,0)$, then it is on the boundary. Hence from the preceding argument, it implies that

$$0 = \int_0^t C^T [\int_{-h}^0 T(t-s) d_s \hat{C}(s-l,l)] \, ds$$

So that $C^T [\int_{-h}^0 T(t-s) d_S \hat{C}(s-l,l)] = 0 \;\; a.e.$

This, by the definition of properness implies that the system is not proper since $C^T \neq 0$. However, if $0 \in \; interior \, R(t,0) \; for \; t > 0$

$$C^T [\int_{-h}^0 T(t-s) d_s \hat{C}(s-l,l)] = 0 \;\; a.e \;\Rightarrow c = 0$$

Which is the properness of the system and by the equivalence in **Theorem 3.1**, the relative controllability of system (3.1) on the interval $[0,a]$; $a > 0$ is thus proved.

**Reference**

Balachandran, K. and Dauer, J.P. (1996) "Local Null Controllability of Non- linear Functional Differential Systems in Bamach Spaces". Journal of optimal Theory and App 88, pp61-75.

Balachandran, K. and J.P.Dauer (1998), "Existence of Solution of a Non-linear Mixed Neutral Equations, Applied Mathematics Letters II No 2, pp 23-28.

Chukwu E. N. (1988), "The Time Optimal Control Theory of Linear Differential Equations of Neutral Type", Journal Computer. Maths and Applications, vol.16 pp 851-866.

Chukwu, E.N. and S.M Lenhart (1991), "Controllability Questions for Non-linear Systems in abstract spaces', Journal of optimization theory and Applications, 18, pp 437-462.

Gyori, I. (1982), "Delay Differential and Integrodifferential Equations in Biological Compartmental Models" Systems Science (Wroclaw) Poland, 8,pp 7-187.

Hale, J. and Kato, J. (1978), "Phase space for retarded equations with infinite delay", Funkcial Ekvac, 21, pp 11-41.

Hino, Y., S. Murakami and T. Naito (1991), "Functional Differential Equations with infinite delay", Lecture Notes in Maths, vol.1473, springer-vorlag, Berlin.

Klanka, J. (1976), Relatively Controllability and Minimum Energy Control of Linear Systems with Distributed Delay in Control", E.E.E. Trans Automatic Control, AC-21.

Naito, K. (1989), Approximate Controllability for Trajectories of Semi-linear Control Systems", Journal of optimum Theory and Applications, 6, pp 57-65.

Yamamoto, M. and J. Park, (1990), "Controllability for Parabolic Equations with uniformly bounded non-linear terms", Journal of optimization theory and Applications, 66, pp 515-532.

Quinn, M. D. and N. Carmichael, (1984-1985), "An Approach to Non-linear Control Problems using Fixed point methods, degree theory and

Pseudo-inverses", Number. Funct. Anal-Optim, 7, pp 197-219.