Integrating Algebra in a Pre-Calculus Subject: A Concrete Approach

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Abstract. Engineering students find difficulties in learning the basics of pre-calculus due to prior weakness in Algebra resulting from the excessive use of calculators in solving algebraic problems. In this paper, we will attempt to make a correlation between teaching algebraic techniques as part of a pre-calculus course and the students' learning and understanding of the different pre-calculus core concepts. The effectiveness of the proposed teaching methodology was tested on a random sample of 100 students in the faculty of Engineering at the American University of Sharjah in the UAE. The students in the hybrid pre-calculus mathematics course with 22 extra hours of algebraic lectures (experimental group) were compared to students that were traditionally lectured (control group). Both groups were assessed. The experimental group exhibited deeper learning of pre-calculus concepts and showed better overall performance compared with the control group who struggled to finish the questions that required core algebraic concepts.

Keywords: Algebra, Pre-Calculus, Learning Outcomes, Mathematics, Higher Education.

1 INTRODUCTION

As Mary Everest Boole stated back in 1909 as part of her Philosophy and Fun of Algebra: "But when we come to the end of our arithmetic we do not content ourselves with guesses; we proceed to algebra **--** that is to say, to dealing logically with the fact of our own ignorance." Algebra is a branch of mathematics that deals with properties of operations and the structures these operations are defined on. What puts elementary algebra a step ahead of elementary arithmetic is a systematic use of letters to denote generic numbers. Mastering of elementary algebra is often hailed as a necessary preparatory step for the study of Calculus. However, the symbolism that is first introduced in elementary algebra permeates all of mathematics and can be considered as the alphabet of the mathematical language.

Elementary algebra can be thought of as the gateway to higher math courses and as can be seen in Figure 1, is divided into multiple parts: Main concepts, Visualizations, and applications.

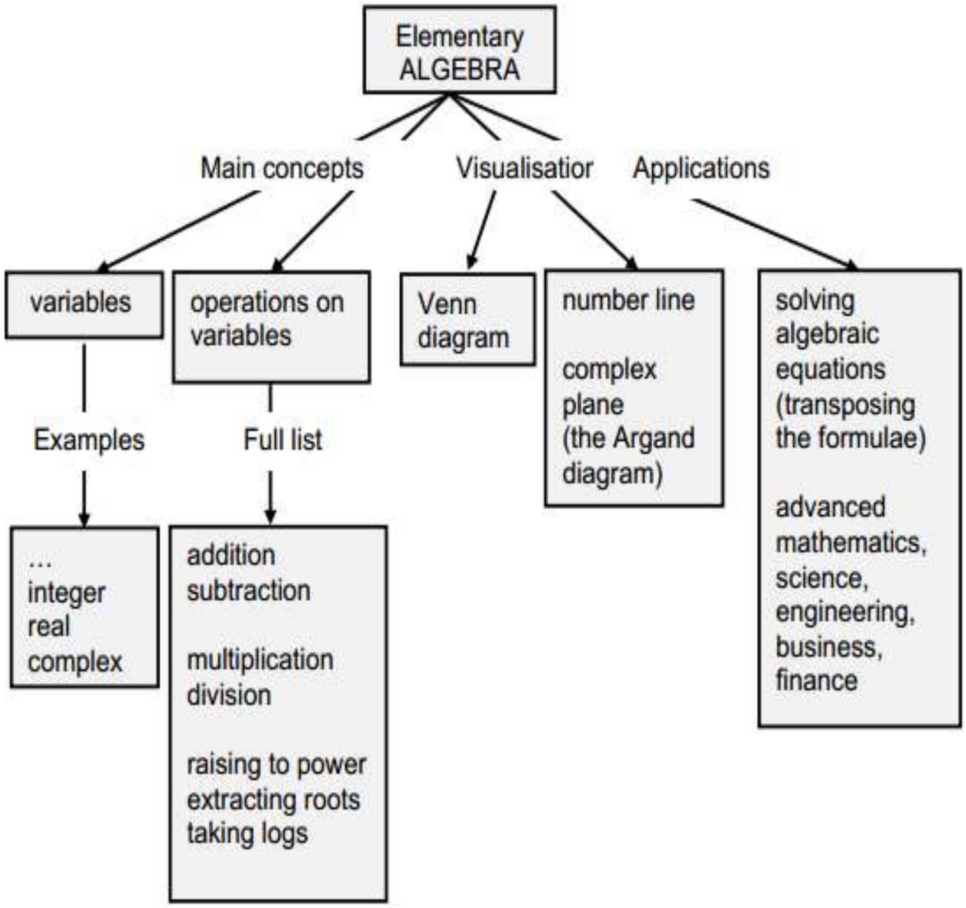

Figure 1. A graphical concept map representing the major components that form Elementary Algebra.

On the other hand, calculus is the study of how things change. It provides a framework for modeling systems in which there is change, and a way to deduce the predictions of such models. With this students get the ability to find the effects of changing conditions on the system being investigated. By studying these, they can learn how to control the system to make it do what they want it to do. Calculus gives engineers the ability to model and control systems in the material world. The development of calculus and its applications to physics and engineering is probably the most significant factor in the development of modern science beyond where it was in the days of Archimedes. And this was responsible for the industrial revolution and everything that has followed from it including almost all the major advances of the last few centuries.

To summarize, calculus provides procedures for solving problems in the analysis of change: determining rates of change, predicting the amount and explaining the quality of change, and connecting the concepts of change with the language and symbolism of algebra that describes change. A typical concept map for calculus concepts is depicted in Figure 2.

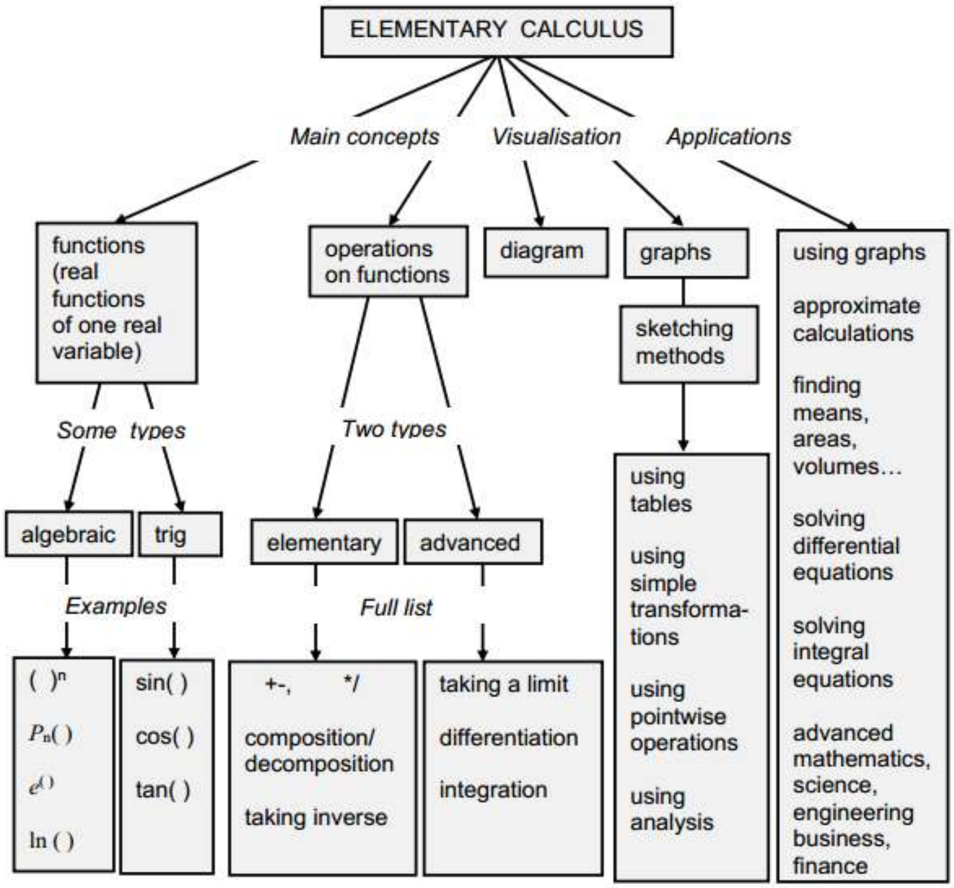

Figure 2. A graphical concept map representing the major components that form Elementary Calculus

In traditional engineering majors in universities, the jump a student makes from learning elementary algebra to learning calculus can be a difficult one for most students especially those coming with a weaker mathematical background and hence the need for pre-calculus subjects that would give background for the mathematical concepts, problems, issues and techniques that appear in the future calculus course.

Certainly one key background tool for calculus is the function concept. Being familiar with function concepts and specific functions provides an important foundation and language for calculus. To understand calculus, students should have a background that allows them to use

- numbers and variables in the context of algebra,
- equations and functions both algebraically and visually, and
- "Real world" applications that use functions to relate the quantities involved.

A sensible approach in pre-calculus to prepare students for calculus includes the following learning milestones:

- Renew the understanding of numbers and variables as used in algebra.

- Renew the understanding of equations both algebraically and visually.
- Expand the understanding of functions both algebraically and visually.
- Connect "real world" applications to equations and functions
- Introduce problems of the type encountered in calculus when pre-calculus techniques can be used for solution.

As can be noted in the above set of pre-calculus milestones, renewing the understanding of several algebraic concepts is part of any pre-calculus course. However, as was noticed over several semesters at the American University of Sharjah, students struggled with basic pre-calculus concepts due to their lack of knowledge in basic algebraic concepts. Also, the time allocated for pre-calculus material as per the syllabus did not have any room for additional concepts during lecture hours. This resulted in the need of some changes in delivering the course to integrate both algebraic core concepts and pre-calculus concepts.

The rest of this paper is organized as follows. In Section 2, existing literature work discussing the need for Algebra and its effects on the delivery of higher math courses is discussed. In Section 3, we introduce our methods in complementing the delivery of a pre-calculus course with a whole semester of Algebra lectures being run in parallel. In Section 4, we present our results from both the sample of 100 students representing the control group (normal delivery) and another sample of 100 students representing the experimental group (added Algebra lectures). We conclude this paper in Section 5.

2 RELATED WORK

Elementary Algebra has been the foundation to understand higher math courses in Engineering. The research literature indicates that declining student participation in mathematics is of broad concern at many levels (Thompson & Fleming, 2003; Barrington, 2006). Students' perception and knowledge of basic algebraic concepts has been a main reason in the evident decline and most importantly failure in Algebra would usually result in the negative attitude towards higher level courses including pre-calculus and calculus (MacGregor, 2004). Algebra acts as an indicator for further study in mathematics (e.g., MacGregor, 2004; Stacey & Chick, 2004). The development of positive attitudes to the subject are essential to increase student success in advanced mathematics subjects and most importantly success in pre-calculus and other intermediate math courses.

Some critical aspects that have led students to be week in algebra include an over-reliance on textbook work with a procedural focus and closed learning activities that result in a lack of understanding and capacity to transfer knowledge (Hollingsworth, Lokan, & McRae, 2003; Gregg 1995; Perry, Howard, & Tracey, 1999).

Also, many students report that they neither understand important algebraic concepts nor appreciate why they are worth the effort of learning (Watt, 2005). In terms of the difficulty students have with algebra study, Stacey and MacGregor (1999) found that only 8% of 116 Year 10 students could solve an equation with variables on both sides if it included fraction operations. Given this lack of success in algebra study it is hardly surprising that many students found algebra hard and also did not know how or when to apply basic algebraic concepts in pre-calculus courses. According to Kaput (1987), the weaknesses in school algebra is the heart of all problems in higher mathematics courses.

Researches have pointed at the role of school algebra in studying higher level math courses and describe algebra as a filter that decides who can go to college and who can't. Stacey and Chick (2004) claim that algebra became the language needed for teaching mathematics. That's why students should be given the opportunity to learn algebra properly. Kaput (1995) claims that school mathematics should be monitored across all levels and all topics, and thus algebrafying the math curriculum at schools. Teachers are central to any student success especially in math related core concepts (Doerr, 2004). Consequently, it is imperative to analyze systematically all aspects of teachers' classroom practices including the intended curriculum guidelines and lesson plans (Taylor, Muller, & Vinjevold, 2003).

The purpose of this paper is to briefly describe an algebra intervention as part of a pre-calculus course in order to improve the understanding of core pre-calculus concepts and therefore result in over better performance in the course.

3 METHOD

Since its introduction to the market more than 40 years ago, the calculator has evolved from being a basic machine capable of doing simple operations (+,-, /,*) to a device that can solve highly complicated algebraic equations in seconds. Despite the obvious benefits for using calculators in mathematics subjects, excessive use of calculators will impair student's mathematical understanding and put limits on student's creativity in solving and analyzing. Calculators should be used as tool to help the students solve problems and not solve the problems themselves. At the American University of Sharjah (AUS), it was noticed that while teaching advanced math problems in pre-calculus, the students understand the concept and the way of work, but they get stuck at the end of the problem as they forget basic algebraic concepts (e.g. how to factor even the simplest quadratic functions; how to do common denominator; or even how to simplify the powers, and how to work with radicals). The pre-calculus course (a 4- credit course with 3-hours lecture and a 2-hours recitation) at AUS was designed to cover the following objectives:

1. Develop the basic properties of real and complex numbers.
2. Solve rational, radical equations and polynomial inequalities.
3. Define the basic concepts of functions, the concepts of domain and range and sketch functions by transformation.
4. Perform basic operations on functions and conclude new domains.
5. Find the inverse of a function, if exists.
6. Sketch the graph of logarithmic and exponential functions and solve equations with exponential and logarithmic expressions.
7. Define some basic trigonometric identities, and solve trigonometric equations.
8. Sketch trigonometric functions and identify domain, periods, amplitudes.

Normally, we offer four weeks (8 hours) of algebraic lectures as part of the recitation hours in the pre-calculus course consisting of a sample of 100 engineering students at the American University of Sharjah. More specifically, as can be seen in Table 1 eight hours of algebraic lectures were introduced during the 2-hours recitation periods so that the lecture schedule would not be affected. The algebra material was taught in 4 recitations (total=8 hours), and the remaining recitations were used to solve practice problems on the material of the course. This first group of 100 students represents our control group in our analysis.

Table 1: The introduction of 4 lectures (2 hours each) during the first four recitations to review basic core algebraic concepts.

Recitation 1	R1-2-3: Sets, Real Numbers, Polynomials
Recitation 2	R4: factoring
Recitation 3	R5: Rational expressions
Recitation 4	R6-7: Rational Exponents, Radical expressions

Also, an assessment in the form of an exam in Week 5 is conducted to test the students on the algebraic concepts (R1-R7). Calculators were not allowed to be used in the algebra exam.
To see the effects of algebra on the study of pre-calculus, we decided to give algebraic lectures during the whole semester to another group of 100 students as part of the recitation hours in the pre-calculus course. More specifically, as can be seen in Table 2, 22 hours of algebraic lectures were introduced during eleven 2-hours recitation periods so that the lecture schedule would not be affected. This second sample of 100 students represents our experimental group.

Table 2: The introduction of 11 lectures (2 hours each) during the whole semester recitations to cover basic core algebraic concepts.

Recitation 1	R1(Sets), R2(Real Numbers and their properties)
Recitation 2	R3(polynomials)+Long division
Recitation 3	R4(factoring)
Recitation 4	Review 1 (factoring)
Recitation 5	R5 (Rational expressions)
Recitation 6	Review 2 (Rational expressions)
Recitation 7	R6 (Rational Exponents)
Recitation 8	R7(Radical expressions)
Recitation 9	Review 3(Radicals)
Recitation 10	Review
Recitation 11	Recitation exam

The students in the hybrid pre-calculus mathematics course with 22 extra hours of algebraic lectures (experimental group) were compared to students that were traditionally lectured as per the pre-calculus syllabus with only eight hours of algebra (control group). Both groups were assessed.

4 ANALYSIS AND RESULTS

4.1 Evident Observations

It was clearly noticed in the control group during the exams that while teaching basic pre-calculus concepts including: solving rational equations, radical equations, polynomial inequalities, solving logarithmic equations, exponential equations, and trigonometric equations, students understand the method of solving these problems, but unfortunately they fail to finish these problems due to their weakness in algebra. The following are examples of questions and student reasoning in the pre-calculus midterm exam showing some clear lack of knowledge in basic algebraic concepts hindering the students reaching the final solution.

Solve the following:

a) $(\log x)^2 - log(x^2) - 8 = 0$

b) $e^{3x} - e^{2x} - 12e^x = 0$

c) $\frac{(e^{3x+1})^2}{e^4} = e^{10x}$

d) $\frac{(5^{2x-1})^3 5^{2x}}{5^{3x} 5^{-x}} = 25$

e) $\frac{x}{x+5} + \frac{2}{x+3} = \frac{1}{x^2+8x+15}$

f) $\frac{x+1}{x-2} + \frac{x-3}{x-1} \geq 0$

g) $\sqrt{x+2} + \sqrt{3x+7} = 1$

h) $t^{\frac{2}{3}} + t^{\frac{1}{3}} - 6 = 0$

i) $(x-1)^{\frac{2}{3}} - 7(x-1)^{\frac{1}{3}} - 8 = 0$

j) $2\sin^2\theta + 7\cos\theta = 5,\ 0 \leq \theta \leq 2\pi$

k) $5\tan^2 x = 3\tan x + \sec^2 x,\ 0 \leq x \leq 2\pi$

A detailed analysis of the above questions on a number of student solutions indicated the following interesting observations regarding questions:

a) Students were able to apply the logarithm properties ($\log\ M^r = rlogM$), and change the problem to a quadratic equation, and then they failed to finish the problem.
b) Students change the problem to a polynomial equation (by putting $u = e^x$) and then did not know how to finish.
c) Students were unable to simplify the powers , they forgot the basic algebraic concepts that state that: $(a^r)^s = a^{rs}; \frac{a^r}{a^s} = a^{r-s}; a^r a^s = a^{r+s}$
d) Similar analysis to question (c).
e) The students fail to do the common denominator, and thus were unable to continue the problem.
f) Similar analysis to question (e).
g) The students were able to isolate one radical and then square both sides, but unfortunately they did not know the algebraic quadratic rule $(A+B)^2 = A^2 + 2AB + B^2$ resulting in a wrong conclusion.
h) The students were unable to notice that $(x-1)^{2/3}\ is\ the\ square\ of\ (x-1)^{1/3}$, and $t^{2/3}\ is\ the\ square\ of\ t^{1/3}$ which was a good indication of lack of knowledge in power basics.
i) Similar analysis to part (h).
j) The students know how to apply the trigonometric identities :

$$sin^2x + cos^2x = 1\ ;\ 1 + tan^2x = sec^2x$$

But unfortunately, the students are unable to factor the equation.

k) Similar analysis to part (j).

4.2 Detailed Analysis

As evident in Figure 3, the failure rate (letter grade F) decreased drastically by about **50%** (from 14% to 7%) while the number of 'A' letter grades quadrupled. This result is not a surprise as using the proposed methodologies will have an impact more on the students who need more help understanding the algebraic concepts and thus decreasing the failure rate but at the same time reinforcing the algebraic concepts especially in the tricky questions that require more algebra and therefore increasing the number of 'A' students.

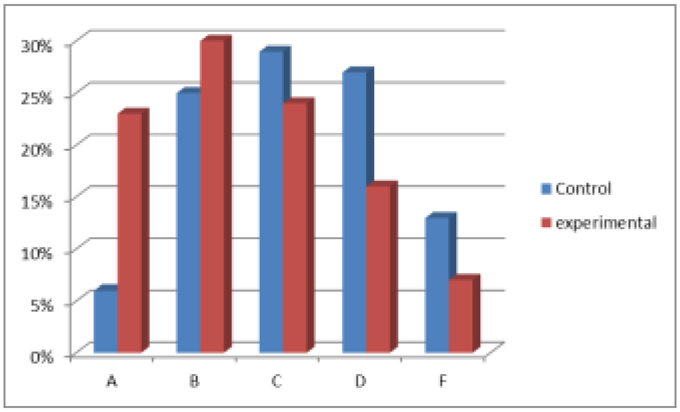

Figure 3. The performance (Letter Grades) of the students before (control group) and after (experimental group) using the proposed teaching methodologies in the pre-calculus course.

As seen in Figure 4, the students in the control group with traditional pre-calculus lecturing and 8 hours of lectures on algebra got an average 64% in the Algebra portion exam while they got an average of 66% in the overall course (excluding chapter R). On the other hand, the students in the experimental group with 11 algebra lectures embedded in the course delivery got an average 75% in the Algebra portion exam while they got an average of 75% in the remaining material of the course (excluding chapter R). This was a clear indication of the evident improvement in the performance in the course given more time is spent in delivering Algebra concepts.

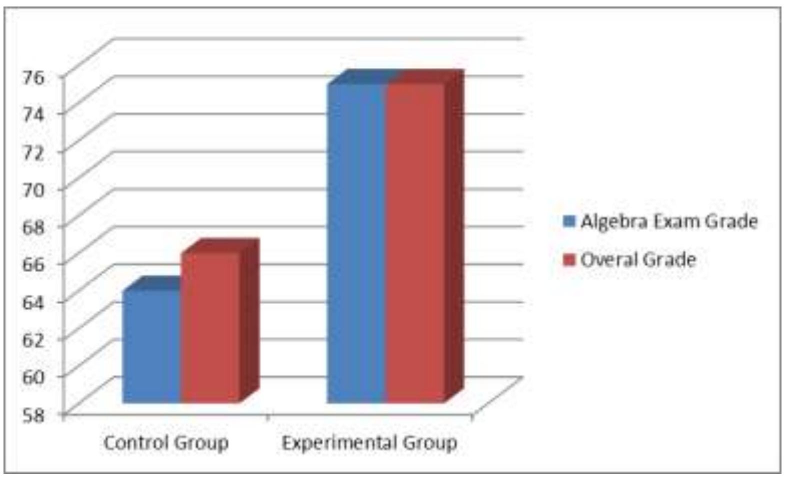

Figure 4. The average grade in the Algebra portion exam and the overall pre-calculus grade in both the control group (normal) and experimental group (more Algebra lectures).

5 CONCLUSIONS

Algebra is, as in most mathematical disciplines, an important tool in calculus, and a good understanding of Algebra and limits is fundamental for understanding calculus. Usually engineering students need to take a pre-calculus course in Mathematics as part of their degree requirements. However, many engineering students find difficulties in learning the basics of pre-calculus due to prior weakness in Algebra. The knowledge of algebra allows for the cultivation of analytical adeptness and proficiency complex thought processes in pre-calculus, rather than just the development of mechanical and computation skills. As evident in our analysis section, reducing the number of hours spent on pre-calculus problems, and using these hours to cover the gaps in algebra for the whole semester will improve students' overall performance in Pre-calculus.

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